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FOUNDATIONS

✱**Wang, Hao, et McNaughton, Robert. Les systèmes axiomatiques de la théorie des ensembles.** Gauthier-Villars, Paris; E. Nauwelaerts, Louvain, 1953. 55 pp. 750 francs; $2.38.

This book contains a brief account of the most important systems of axiomatic set theory, results concerning their relative consistency and a very good up-to-date bibliography of the subject. It forms a compact survey of the various axiom systems. It is divided into seven chapters.

Chapter I deals with Cantor's naive set theory which led up to the Russell paradox. The next four chapters describe various systems of set-theory which, so far at least, have not been proved inconsistent. In chapter II we have the theory of types. Chapter III describes Zermelo set-theory and the Zermelo-Fraenkel system (without the axiom of choice), and chapter IV its extension to the von Neumann-Bernays system. Chapter V describes Quine's system of New Foundations [Amer. Math. Monthly **44**, 70–80 (1937)]. Due to the difficulty of proving a statement equivalent to the axiom of infinity in this system, it seemed unsuitable for the development of number theory. It was therefore enlarged to the system of "Mathematical logic" [Norton, New York, 1940, rev. ed., Harvard Univ. Press, 1951; these Rev. **2**, 65; **13**, 613]. However, since this book on the systems of set theory was written, E. Specker has succeeded in proving the axiom of infinity in the system of New Foundations [see the paper reviewed below; in this paper he has also disproved the general axiom of choice in New Foundations]. Chapter VI describes a sequence of systems $T_1, T_2, \cdots$, where T_1 is Zermelo's system without the axiom of infinity, and T_n is essentially the theory of types up to the nth type with the sets of T_1 as the set of individuals. For any $n \geqq 1$, it is possible to prove the consistency of T_n in T_{n+1}, so that the systems increase in strength with n. Chapter VII contains a collection of results about the relative strength of the various systems, for example: (1) a proof of the consistency of the system of Mathematical Logic relative to that of New Foundations can be formalized in T_3; (2) a proof of the consistency of the von Neumann-Bernays system relative to the Zermelo-Fraenkel system can be formalized in T_2 [R. McNaughton, J. Symbolic Logic **18**, 136–144 (1953); these Rev. **14**, 1052]; (3) every theorem of the von Neumann-Bernays system which can be expressed in the symbolism of the Zermelo-Fraenkel system is a theorem of the Zermelo-Fraenkel system [A. Mostowski, Fund. Math. **37**, 111–124 (1950); these Rev. **12**, 791]; (4) the consistency of the theory of types can be proved in Zermelo's system [J. Kemeny, Dissertation, Princeton, 1949]. *I. Novak Gál.*

Specker, Ernst P. The axiom of choice in Quine's New Foundations for Mathematical Logic. Proc. Nat. Acad. Sci. U. S. A. **39**, 972–975 (1953).

Specker here proves the following important results which were long a subject for conjecture: 1) the axiom of infinity is provable in Quine's "New Foundations" [Amer. Math. Monthly **44**, 70–80 (1937)]; 2) the axiom of choice is disprovable in "New Foundations"; 3) the generalized continuum hypothesis is also disprovable in "New Foundations". (The proof of 3) is only sketched, but it is possible to carry it out in New Foundations.) *I. Novak Gál.*

Wang, Hao. The categoricity question of certain grand logics. Math. Z. **59**, 47–56 (1953).

The author takes a system S and defines a denumerable model of S within S and shows that one can consistently add an additional axiom by which the objects of S are restricted to that domain. The system S he uses is Quine's "New Foundations" [Amer. Math. Monthly **44**, 70–80 (1937)] to which the axiom of infinity and the axiom of choice are added. This system was, however, recently proved inconsistent by Specker [see the preceding review], so that it is not clear how many of Wang's results still hold or can be carried over to other systems. *I. Novak Gál.*

✱**Péter, Rózsa. Transfinite Rekursionen (Grundlagenforschung und rekursive Funktionen).** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 419–428. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian and German summaries)

This is a review of the history of the theory of recursive functions from the recursive number theory of Skolem [Skr. Vid. I. Kristiania. Mat.-Nat. Kl. **1923**, no. 6] to the work of Gödel, Ackermann, Herbrand, Kleene, Church, Markov, Post, Kálmár and Péter. The newest result mentioned states that transfinite recursion of type ω^ω leads out of the class of multiply recursive functions [Péter, J. Symbolic Logic **15**, 248–272 (1950); these Rev. **12**, 469]. The author stresses that the existence of undecidable problems in mathematics is due to the limitations imposed by the definition of "effective process." (This review is based on the German summary.) *I. Novak Gál* (Ithaca, N. Y.).

Simons, Leo. New axiomatizations of S3 and S4. J. Symbolic Logic **18** (1953), 309–316 (1954).

A system of six axiom schemata is given which, together with the rule of detachment for material implication, defines Lewis' S3. By the addition of $\Diamond\Diamond a \rightarrow \Diamond a$ as an axiom schema, an axiomatization of S4 is obtained. The independence of the axioms is proved. *A. Heyting* (Amsterdam).

✱**Johansson, Ingebrigt. On the possibility to use the subtractive calculus for the formalization of constructive theories.** Actes du XIème Congrès International de Philosophie, Bruxelles, 20–26 Août 1953, vol. XIV, pp. 60–64. North-Holland Publishing Co., Amsterdam; Editions E. Nauwelaerts, Louvain, 1953.

The author suggests that the subtractive calculus, formally dual to the positive calculus, also be used to deal with constructive theories. The reason is primarily that

every metatheoretical proposition of the form

" '$(\forall x)(f(x) \supset g(x))$' is true"

must in this calculus be replaced by

" '$(\exists x)(f(x) - g(x))$' is false",

thus implying the impossibility of certain constructions. ('$f(x) - g(x)$' is to be read '$f(x)$ without $g(x)$'.)

I. Novak Gál (Ithaca, N. Y.).

Ohnishi, Masao. On intuitionistic functional calculus. Osaka Math. J. **5**, 203–209 (1953).

Let F be a closed formula of the lower predicate calculus in prenex form with a prefix consisting of two or three alternating general and existential quantifiers. The author investigates the deducibility relations which exist in the intuitionistic logic between the propositions resulting from F by the insertion, in all possible ways, of double negations before, between, or after the quantifiers. As an application he gives the three possible definitions of a convergent sequence of rational numbers and examines the relations between the different notions of equality and inequality between such sequences. For positively convergent sequences the three notions defined by Brouwer [Jber. Deutsch. Math. Verein. **33**, 251–256 (1925)] result.

A. Heyting (Amsterdam).

Freudenthal, Hans. Meaningful interpretations in formal logic. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde **62**, 94–96 (1953). (Dutch)

The author proposes to solve the paradoxes of material implication by defining a "full implication" as a relation between two propositions for which the three combinations of truth-values 00, 01, 11 are possible (0 = false, 1 = true). The paradoxes arise in cases where one of two of these combinations are excluded a priori, so that the implication is not a full implication. *A. Heyting* (Amsterdam).

Gottschalk, W. H. The theory of quaternality. J. Symbolic Logic **18**, 193–196 (1953).

The concept of duality, and the square of opposition of classical logic, are subsumed under a general theory of quaternality. If A and B are classes (in particular, algebras) with involutions α and β, the group of quaternality is the four-group carrying each operator f from A to B into f, $f\alpha$, βf, and $\beta f\alpha$. Special cases: $A = B$, $\alpha = \beta$; and α (or β) the identity. Applications: the lower predicate calculus, this with quantifiers, with equality, with modal operators; the calculus of classes, of relations. *R. C. Lyndon.*

Cassina, Ugo. Sulla critica di Grandjot all'aritmetica di Peano. Boll. Un. Mat. Ital. (3) **8**, 442–447 (1953).

L'autore espone e confuta la critica di Grandjot all'aritmetica di Peano. Dà una nuova dimostrazione dell'esistenza ed unicità della somma di due numeri, che ritiene più semplice di quella di Kálmár, esposta da Landau, per ovviare alla critica di Grandjot. Termina con alcune considerazioni sul principio d'induzione. *Author's summary.*

Moise, E. E. The use of induction in existence proofs. Amer. Math. Monthly **61**, 192–193 (1954).

* **Jaffé, George. Drei Dialoge über Raum, Zeit und Kausalität.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1954. 211 pp. DM 9.60.

ALGEBRA

Hall, Marshall, Jr., and Connor, W. S. An embedding theorem for balanced incomplete block designs. Canadian J. Math. **6**, 35–41 (1954).

If one block and all the varieties in it are deleted from a symmetrical incomplete balanced block design with parameters v, k, and λ one obtains the derived design with parameters $v-k$, $v-1$, k, $k-\lambda$, and λ. It is however known that it is not always possible to integrate a design with these parameters and to obtain a symmetrical design from it through the adjunction of k varieties and one block. A counter example with $\lambda = 3$ has been constructed by Bhattacharya [Science and Culture **9**, 508 (1944); these Rev. **6**, 163]. On the other hand it is easy to see that integration is always possible if $\lambda = 1$. The authors prove that integration is always possible also when $\lambda = 2$. The proof depends on methods developed by W. S. Connor [Ann. Math. Statistics **23**, 57–71 (1952); **24**, 135 (1953); these Rev. **13**, 617].

H. B. Mann (Columbus, Ohio).

Nair, K. R. A note on group divisible incomplete block designs. Calcutta Statist. Assoc. Bull. **5**, 30–35 (1953).

After an exposition of recent results, obtained by R. C. Bose, W. S. Connor, Shimamoto, Rao and the author, the author discusses relations between group-divisible designs and certain special types of confounded two-factor designs.

H. B. Mann (Columbus, Ohio).

Clatworthy, Willard H. A geometrical configuration which is a partially balanced incomplete block design. Proc. Amer. Math. Soc. **5**, 47–55 (1954).

Let (A_{ij}) be a skew symmetric matrix with elements from $GF(s)$. The points $\theta = (1_1, 1_2, 1_3, 1_4)$ and ϕ are called first associates if ϕ lies on the plane $\pi(\theta)$ whose points satisfy the equation $\sum_i \sum_j a_{ij} x_i x_j = 0$. Otherwise they are called second associates. Form blocks consist of the lines through θ lying in $\pi(\theta)$. The author shows that these blocks form a PBIB with parameters $v = b = (s^4-1)(s-1)$, $r = k = s+1$, $\lambda_1 = 1$, $\lambda_2 = 0$, $n_1 = s^3+s$, $n_2 = s^3$, $p^1_{11} = s-1$, $p^2_{11} = s+1$, $p^1_{12} = s^2$, $p^1_{22} = s^2(s-1)$, $p^2_{12} = s^2-1$, $p^2_{22} = s^2(s-1)$. The design for $s = 3$ is explicitly given in the paper.

H. B. Mann (Columbus, Ohio).

Connor, W. S., and Clatworthy, W. H. Some theorems for partially balanced designs. Ann. Math. Statistics **25**, 100–112 (1954).

The customary notation for the parameters of a PBIB will be used in what follows [see, e.g., Bose and Shimamoto, J. Amer. Statist. Assoc. **47**, 151–184 (1952); these Rev. **14**, 67]. Put $a_{su} = \lambda_1 p^u_{s1} + \cdots + \lambda_m p^u_{sm} - \lambda_s n_s$ for $s \neq u$ and $a_{ss} = z + \lambda_1 p^s_{s1} + \cdots + \lambda_m p^s_{sm} - \lambda_s n_s$. Let A be the matrix (a_{su}), $z_1, \cdots, z_t$ the distinct roots of the equation $|A| = 0$, $\alpha_1, \cdots, \alpha_t$ resp. their multiplicities and $z_0 = -n_1\lambda_1 \cdots - n_m\lambda_m$. Let N be the incidence matrix of a PBIB and form the matrix NN'. Let M be the matrix obtained by replacing in NN' the parameter r by the variable z. The authors

show that $|M|=(s-s_0)(s-s_1)^{a_1}\cdots(s-s_t)^{a_t}$ and hence $NN'=rk(r-s_1)^{a_1}\cdots(r-s_t)^{a_t}$. Since $b\geqq$ rank M one obtains $b>v$ for $r\neq s_u$, $u=1, \cdots, t$, and $b>v-a_u$ for $r=s_u$. For a resolvable design one obtains $b>v+r-1$ if $r\neq s_u$, $u=1, \cdots, t$ and $b>v+r-1-a_u$ if $r=s_u$. Since NN' is positive semi-definite one also obtains $r>s_u$. The authors apply their result further to group divisible and triangular designs and to designs with 2, 3 and 4 associate classes and obtain several new necessary conditions for the existence of such designs with given parameters. Their results are too numerous and too complicated to be enumerated in a review.

H. B. Mann (Columbus, Ohio).

Shrikhande, S. S. Cyclic solutions of symmetrical group divisible designs. Calcutta Statist. Assoc. Bull. **5**, 36–39 (1953).

Symmetrical group divisible designs have been defined by R. C. Bose and W. S. Connor [Ann. Math. Statistics **23**, 367–383 (1952); these Rev. **14**, 124]. Their notation and terminology is used in what follows. The author proves the following theorem. Theorem I: let a be an odd positive factor of m. If the symmetrical group-divisible design with $K^2-\lambda_2\nu>0$ has a cyclic solution, then the diophantine equation $X^2=(K^2-\lambda_2\nu)Y^2+(-1)^{\frac{1}{2}(a-1)}aZ^2$ must possess a solution in integers not all zero. The author also states the following theorem. Theorem II: Let c be an odd positive factor of n, $(m, c)=1$. If the symmetrical group-divisible design with $K-\lambda_1>0$, has a cyclic solution, then the diophantine equation $X^2=(K-\lambda_1)Y^2+(-1)^{\frac{1}{2}(c-1)}cZ^2$ has a nontrivial solution in integers X, Y, Z. The proof of Theorem I utilizes a result of Hall and Ryser [Canadian J. Math. **3**, 495–502 (1951); these Rev. **13**, 312]. The proof of Theorem II is not explicitly given, but runs along lines similar to that of Theorem I.

H. B. Mann (Columbus, Ohio).

Seiden, Esther. On the problem of construction of orthogonal arrays. Ann. Math. Statistics **25**, 151–156 (1954).

Generalizing a method of R. C. Bose [Proc. Internat. Congress Math., Cambridge, Mass., 1950, v. 1, Amer. Math. Soc., Providence, R. I., 1952, pp. 543–548; these Rev. **13**, 522] for the construction of orthogonal arrays of strength 2, the author gives a new method for the construction of orthogonal arrays of strength t. Using her method the author utilizes the two level multifactorial designs of Plackett and Burman [Biometrika **33**, 305–325 (1946); these Rev. **8**, 44] to construct orthogonal arrays of strength 3 with the highest possible number of rows. The author also constructs an orthogonal array (36, 13, 3, 2) and shows that her method in this case cannot lead to arrays with more than 13 constraints.

H. B. Mann (Columbus, Ohio).

Shapiro, V. L. A theorem on a special class of near-Vandermonde determinants. Amer. Math. Monthly **60**, 697–699 (1953).

The determinant $\Delta_n=|a_{ij}|$, where $a_{1j}=j$ and $a_{ij}=j^{2(i-1)}$ $(i=2, 3, \cdots, n;\ j=1, 2, 3, \cdots, n)$, is nonzero when $n=\frac{1}{2}(p+1)$ and p is an odd prime. Determinants of this type occur in some work on trigonometric series.

W. Ledermann (Manchester).

Rutherford, D. E. Some continuant determinants arising in physics and chemistry. II. Proc. Roy. Soc. Edinburgh. Sect. A. **63**, 232–241 (1952).

Let $P_n(x)$ denote the square matrix

$$\begin{Vmatrix} x & 1 & 0 & \cdot & \cdots & \cdot & \cdot \\ 1 & x & 1 & \cdot & \cdots & \cdot & \cdot \\ 0 & 1 & x & 1 & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & 1 & x \end{Vmatrix}$$

of order n, and let $\varphi_n(x)$ denote its (continuant) determinant. In an earlier paper [same Proc. **62**, 229–236 (1947); these Rev. **8**, 499] the author evaluated and factored a number of determinants related to $\varphi_n(x)$. In this paper he begins by listing the more important formulas which evaluate $\varphi_n(x)$; it is amusing that $i^n\varphi_n(-i)$ is the $(n+1)$th Fibonacci number. The evaluations of $\varphi_n(x)$ and some related (circulant) determinants are used to factor certain other determinants and to determine the latent roots of their matrices. For example, the product $P_n(x)P_n(y)$ is a matrix $T_n(s, a)$, where s and a are functions of x and y. The known results for $\varphi_n(x)$ give evaluations of the determinant $|T_n(s, a)|$ and the latent roots of $T_n(s, a)$. The author then calculates the inverses of certain matrices related to $P_n(x)$ and $\varphi_n(x)$. These results are likely to have applications to the solution of systems of linear equations which arise in relaxation problems, or in determining the latent roots of matrices which occur in problems in applied mathematics.

G. B. Price (Lawrence, Kan.).

Berger, W. J., and Saibel, Edward. On the inversion of continuant matrices. J. Franklin Inst. **256**, 249–253 (1953).

A continuant matrix A is one with $a_{ij}=0$ for $|i-j|>1$. A recursive formula is given for the inverse of such a matrix. Its proof begins by reducing A to triangular form. The authors say an n-by-n matrix G has "gnomonic symmetry of type 1" if $g_{in}=g_{i,n-1}=\cdots=g_{ii}=g_{i+1,i}=\cdots=g_{ni}$, for all $i=1, 2, \cdots, n$. Also, G has "gnonomic symmetry of type 2" if reversing the order of both rows and columns results in gnomonic symmetry of type 1. Conditions are given for the inverse of a continuant matrix to have gnomonic symmetry.

G. E. Forsythe.

de Bruijn, N. G., and van Dantzig, D. Inequalities concerning determinants and systems of linear equations. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 315–321 (1952).

Let $P=(p_{ij})$ $(i=1, \cdots, m;\ j=1, \cdots, n)$ be a real or complex rectangular matrix, and let $p_1, \cdots, p_n$ denote its columns. The norm $\|P\|$ of P is defined by

$$\|P\|=\max_{1\leqq i\leqq n}\sum_{i=1}^{m}|p_{ij}|.$$

A vector is considered to be a matrix with one column. The authors establish the properties of the norm $\|P\|$. If P is a square matrix, then P^* denotes its adjoint matrix: if $Q=P^*$, then q_{ij} is the algebraic complement of p_{ji}. The authors establish the following theorem among others: Let P be a real or complex square matrix of order n, and let Q be its adjoint. Furthermore, assume $n\geqq 2$, $\|p_1\|\leqq u_1, \cdots, \|p_n\|\leqq u_n$ and $0\leqq u_1\leqq\cdots\leqq u_n$. Then

$$\|Q\|=\max_{1\leqq j\leqq n}\|q_j\|\leqq(u_1+u_2)u_3u_4\cdots u_n,$$

$$\sum_{1\leqq j\leqq n}\|q_j\|=\sum_i\sum_j|q_{ij}|\leqq u_1\cdots u_n\left(\frac{1}{u_1}+\cdots+\frac{1}{u_n}\right).$$

The bounds are the best possible. If $\|P\| \leqq 1$, then $\|Q\| \leqq \min(2, n)$ and $\sum_i \sum_j |q_{ij}| \leqq n$. These results and others are used to "give extensions of the fundamental theorems of linear algebra which, roughly speaking, consist in replacing the relation '$=0$' by 'sufficiently small' and '$\neq 0$' by 'not too small.'" Definition: If $A \geqq 0$, $\epsilon > 0$, then the integer r is called an (A, ϵ)-rank of a rectangular matrix if the latter contains a square matrix of order r whose determinant has absolute value equal to or greater than A, whereas the determinant of each square matrix of order $r+1$ contained in it is equal to or less than ϵ in absolute value. A typical result established by the authors is the following theorem: Let $P = (p_{ij})$ $(i = 1, \cdots, m;\ j = 1, \cdots, n)$ be a matrix of m rows and n columns, and let U be the $m \times (n+1)$ matrix arising from P by adding an $(n+1)$st column consisting of the numbers $y_1, \cdots, y_m$. If $A > 0$, and if r is an (A, ϵ)-rank of both P and U, then there exists a vector x such that $\|Px - y\| \leqq (m-r)A^{-1}\epsilon$, and $\|x\| \leqq \mu \|P\|^{r-1} \cdot \|y\| \cdot A^{-1}$ $(\mu = \min(2, r))$. The results of the paper have applications to the approximate solution of systems of linear equations.

G. B. Price (Lawrence, Kansas).

Brenner, J. L. Une borne pour un déterminant avec diagonale majorante. C. R. Acad. Sci. Paris **238**, 555–556 (1954).

The following bound for determinants with dominant principal diagonal is obtained:

$$\det|a_{ik}| \geqq |a_{11}| \prod_{k=2}^{n} (|a_{kk}| - \sum_{j<k} |a_{kj}| + L_k)$$

when L_k are certain positive quantities. This improves Price's bound [Proc. Amer. Math. Soc. **2**, 497–502 (1951); these Rev. **12**, 793] where L_k is replaced by zero. Price's bound again improved one of Ostrowski's bounds when "$j<k$" was replaced by "$j \neq k$" [Bull. Sci. Math. (2) **61**, 19–32 (1937)]. The proof is based on the fact [already noticed by P. Furtwängler, Akad. Wiss. Wien, S.-B. IIa. **145**, 527–528 (1936)] that another determinant with dominant principal diagonal is obtained if suitable multiples of the first column are added to the others so as to annihilate all elements of the first row apart from the first one.

O. Taussky-Todd (Washington, D. C.).

***Brauer, Alfred. Bounds for characteristic roots of matrices.** Simultaneous linear equations and the determination of eigenvalues, pp. 101–106. National Bureau of Standards Applied Mathematics Series, No. 29. U. S. Government Printing Office, Washington, D. C., 1953. $1.50.

The main results have now been published elsewhere, mainly in a sequence of five papers by the same author [for the fifth paper and references cf. Duke Math. J. **19**, 553–562 (1952); these Rev. **14**, 836]. *W. Givens.*

Brauer, Alfred. Über die Lage der charakteristischen Wurzeln einer Matrix. J. Reine Angew. Math. **192**, 113–116 (1953).

[Cf. the preceding review.] Certain ovals of Cassini previously shown to contain the characteristic roots of a matrix are improved to $|D_{\kappa\lambda}| \leqq Q_{\kappa\lambda}$, where $D_{\kappa\lambda}$ is the appropriate principal minor of order two of $zI - A$,

$$Q_{\kappa\lambda} = \sum_{\nu} \{|a_{\kappa\lambda}a_{\lambda\nu}| + |a_{\lambda\kappa}a_{\kappa\nu}| + |a_{\nu\kappa}a_{\lambda\nu}|\} + \sum_{\nu<\kappa} |a_{\nu\kappa}a_{\lambda\mu} + a_{\lambda\nu}a_{\kappa\mu}|,$$

and μ and ν vary from 1 to n omitting κ and $\lambda \neq \kappa$. In the notation of part V [cited in the preceding review], $Q_{\kappa\lambda} = P_{\kappa\lambda} - |a_{\kappa\lambda}a_{\lambda\kappa}|$ and it is shown that the new ovals are properly contained in the old. Various other earlier results are similarly improved. *W. Givens* (New York, N. Y.).

***Wielandt, H. Inclusion theorems for eigenvalues.** Simultaneous linear equations and the determination of eigenvalues, pp. 75–78. National Bureau of Standards Applied Mathematics Series, No. 29. U. S. Government Printing Office, Washington, D. C., 1953. $1.50.

The question "what restrictions are imposed upon the eigenvalues λ_i of an n-rowed matrix A by postulating $Ax = y$?" is here studied with a view to obtaining a systematic inclusion theory. Working over the complex field and simplifying by taking x of length one $(\|x\|^2 = x^*x = 1)$ and y nonproportional to x, the following results are obtained. 1. If A is unrestricted, so are the λ_i. 2. If A is Hermitian, define $\mu = x^*y$ and a positive r by $r^2 = \|y\|^2 - \mu^2$. Call a set of real λ_i compatible if some Hermitian A with eigenvalues λ_i carries x into y and call any subset of the λ-axis an inclusion set if it contains at least one point out of every compatible spectrum. Projecting the λ-axis stereographically on the circle C with center μ and radius r, the inclusion sets are the inverse images of sets of C containing a (closed) semicircle. 3. For normal matrices, μ becomes complex, $r^2 = \|y\|^2 - |\mu|^2$, C is replaced by the sphere S with center μ and radius r and semicircles are replaced by hemispheres. 4. For A normalizable (i.e., similar to a diagonal matrix) introduce (i) the real angle $\theta(a, b)$ between any two vectors by $\theta(a, b) = \text{Arc cos}\,(|a^*b| \cdot \|a\|^{-1} \cdot \|b\|^{-1})$ with $0 \leqq \theta \leqq \pi/2$, (ii) the deformation angle $\delta_U = \min \theta(Ux, Uy)$ for all $x^*y \neq 0$, and (iii) the normality angle $\nu_A = \max \delta_U$ for all U such that UAU^{-1} is diagonal. Then $0 < \nu_A \leqq \pi/2$ with equality if and only if A is normal. A set λ_i, is a ν-compatible spectrum if a normalizable A exists with roots λ_i, $Ax = y$ and $\nu_A \geqq \nu$. Then, generalizing 3, the minimal inclusion sets with respect to ν-compatible spectra are the inverse images of spherical segments with spherical radius $\pi - \nu$.

Generalizations to arbitrary A, to more than two vectors and to inclusion theorems for eigenvectors are sketched but cannot be abbreviated here. Proofs are not given.

W. Givens (New York, N. Y.).

Hoffman, Alan J., and Taussky, Olga. A characterization of normal matrices. J. Research Nat. Bur. Standards **52**, 17–19 (1954).

"If $\alpha_1, \cdots, \alpha_n$ are the eigenvalues of an $n \times n$ matrix A, α and β are two complex numbers different from zero, and the eigenvalues of $\alpha A + \beta A^*$ are $\alpha\alpha_i + \beta\bar{\alpha}_{P_i}$, where P is a permutation of the integers $1, \cdots, n$, then A is normal." The same result holds if AA^* has eigenvalues $\alpha_i\alpha_{P_i}$ or if $a_1A + a_2A^* + b_1A^2 + b_2A^{*2} + c_1AA^* + c_2A^*A$, has eigenvalues given by the appropriate polynomial in α_i and α_{P_i}, with $c_1 + c_2 \neq 0$ and a_1, a_2, b_1, b_2, c_1 and c_2 otherwise arbitrary complex numbers. For $c_1 + c_2 = 0$ complicated results are obtained. Proofs depend on the unitary reduction of A to triangular form. *W. Givens* (New York, N. Y.).

Debreu, Gerard, and Herstein, I. N. Nonnegative square matrices. Econometrica **21**, 597–607 (1953).

This is a compact ab initio derivation of the main known properties of n-by-n real matrices $A = (a_{ij})$ satisfying $a_{ij} \geqq 0$. There is a twenty-six item bibliography oriented toward the economic literature. Results not identical with those of Wielandt [Math. Z. **52**, 642–648 (1950); these Rev. **11**, 710] include: for a nonnegative indecomposable matrix A,

$(sI-A)^{-1}$ has all positive elements if and only if s is greater than the maximum (real and positive) root of A; the same is true for the principal minors of $sI-A$ of orders $1, 2, \cdots, n$. There is a confusing misprint in the second of three displayed lines on page 597 where the first $\leqq$ should be the symbol $\leqslant$ which is being defined and distinguished from $\leqq$.

W. Givens (New York, N. Y.).

Drazin, M. P. A note on skew-symmetric matrices. Math. Gaz. **36**, 253–255 (1952).

The author establishes the following two theorems. Theorem 1. Given any even integer n, then there is a polynomial $q(\lambda)=q(a_{ij}, b_{ij}, \lambda)$ $(i, j=1, \cdots, n)$ in the $2n^2+1$ variables a_{ij}, b_{ij}, λ, with integer coefficients, such that if both the $n\times n$ matrices $A=(a_{ij})$, $B=(b_{ij})$ are skew symmetric, then $\det(AB-\lambda I)=\{q(\lambda)\}^2$. Theorem 2. Given any odd integer n, then there is a polynomial $r(\lambda)=r(a_{ij}, b_{ij}, \lambda)$ in the $2n^2+1$ variables a_{ij}, b_{ij}, λ, with integer coefficients, such that, if both the $n\times n$ matrices $A=(a_{ij})$, $B=(b_{ij})$ are skew symmetric, then $\det(AB-\lambda I)=-\lambda\{r(\lambda)\}^2$.

G. B. Price (Lawrence, Kan.).

Kippenhahn, Rudolf. Über den Wertevorrat einer Matrix. Math. Nachr. **6**, 193–228 (1951).

Let $A=(a_{\mu\nu})$ $(\mu, \nu=1, \cdots, n)$ be a square matrix with complex elements, and let the vector $\mathfrak{x}$ be treated as a matrix with one column. Let A^* denote the transposed conjugate of A. The field of values $\mathfrak{W}(A)$ of the matrix A is the set of numbers $\mathfrak{x}^*A\mathfrak{x}$, where $\mathfrak{x}^*\mathfrak{x}=1$. The author investigates the geometrical properties of $\mathfrak{W}(A)$. It is known that $\mathfrak{W}(A)$ is a unitary invariant, and that it is a closed convex set [see Toeplitz, Math. Z. **2**, 187–197 (1918); Hausdorff, ibid. **3**, 314–316 (1919)]. The characteristic roots of A belong to the field of values of A. If H is a Hermitian matrix, $\mathfrak{W}(H)$ is a segment of the real axis bounded by the largest and smallest characteristic roots of H. If A is a normal matrix, then $\mathfrak{W}(A)$ is the convex hull of the characteristic roots of A. The author shows that to each matrix A of the nth order there corresponds a curve $\mathfrak{K}(A)$ of the nth class, whose equation can be given explicitly in terms of A, and whose convex hull is $\mathfrak{W}(A)$. The author investigates the geometrical properties of $\mathfrak{K}(A)$ in great detail. He determines all curves $\mathfrak{K}(A)$ which arise from matrices A of orders 2 and 3. There are two types of curves $\mathfrak{K}(A)$ if A is of order 2: (a) if A is normal, $\mathfrak{K}(A)$ consists of the two characteristic values of A; (b) in all other cases $\mathfrak{K}(A)$ consists of an ellipse whose foci are the two characteristic values of A. If A is of order 3, $\mathfrak{K}(A)$ is a curve of one of four types which are explicitly given. The author obtains inequalities for the diameter, thickness, area, and length of the boundary of $\mathfrak{W}(A)$. Finally, the author extends many of his results to matrices whose elements are quaternions. *G. B. Price.*

Herstein, I. N. A note on primitive matrices. Amer. Math. Monthly **61**, 18–20 (1954).

An algebraic proof is given for the Frobenius theorem that some power of a nonnegative matrix A is positive if and only if A is primitive (i.e., is indecomposable and with a single root of maximal absolute value). The lemmas used are: (i) A primitive implies A^m primitive; (ii) if $\epsilon>0$ and A is nonnegative indecomposable, then $(\epsilon I+A)^{n-1}>0$; (iii) from (ii), $A^{n-1}>0$ if also $a_{ii}>0$ for each i; and, (iv) for a suitable power $m=m(i, j)$, the (i, j) element of A^m is positive if A is nonnegative indecomposable. *W. Givens.*

Mitchell, B. E. Unitary transformations. Canadian J. Math. **6**, 69–72 (1954).

It is well known that if A is a matrix with complex elements, then A is unitarily equivalent to a triangular matrix B whose diagonal elements are the eigenvalues of A; but for given A the matrix B is by no means unique. The author's object is to define, using certain conventions, a canonical form B of A such that two matrices are unitarily equivalent if and only if their canonical forms are identical. Such a canonical form is described when A is non-derogatory. In the derogatory case, the discussion is not yet complete. *D. E. Rutherford* (St. Andrews).

Thurston, H. S., and Alexander, Mary K. The equation $X^2+PX+Q=0$ in binary matrices. Amer. Math. Monthly **61**, 8–17 (1954).

J. J. Sylvester shewed [Philos. Mag. (5) **18**, 454–458 (1884); C. R. Acad. Sci. Paris **99**, 555–558, 621–631 (1884)] that in general, X, P, Q being 2×2 matrices, there are six solutions of $f(X)=X^2+PX+Q=0$, which are determined by means of a cubic equation $\phi(\lambda)=0$ whose coefficients are certain scalar functions of the matrix elements of P and Q. The present authors elucidate the so-called irregular case in which $\phi(0)=0$, a case not fully examined by Sylvester. Each root of $\phi(\lambda)=0$ determines two pairs of scalars (b, d) and each pair determines a solution $X=(P-bI)^{-1}(dI-Q)$ provided $P-bI$ is nonsingular. However, if the pair (b, d) arises from a zero root of $\phi(\lambda)=0$, then $P-bI$ is singular and this pair may lead to one solution of $f(X)=0$, or to a parametric solution, or to no solution at all. The case $P=\mu P+\nu I$ is irregular and is fully analysed in terms of $\phi(\lambda)$ and a certain quartic polynomial. Other irregular cases are examined. *D. E. Rutherford* (St. Andrews).

***Pickert Günter. Lineare Algebra.** Enzyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. I 1, 6. Band I. Algebra und Zahlentheorie. 1. Teil. B. Algebra. Heft 3, Teil I, pp. 1–43. B. G. Teubner Verlagsgesellschaft, Leipzig, 1953. DM 7.50.

Observing that the literature up to 1932 is extensively covered in earlier monographs, the author confines his attention largely to later references. The following are the section headings. A. Modules and vector spaces. Fundamental concepts, linear dependence, vector spaces, linear forms, duality in vector spaces, linear equations, change of field of scalars, matrices. B. Tensor and exterior products. Tensor products of modules, tensors, exterior products of modules, determinants, duality in exterior algebra. The report makes it clear that the concept of a module over a general ring has taken over the central role in modern linear algebra. The fact that half the report is devoted to multilinear algebra is noteworthy. Extensive quotations from Bourbaki [Algèbre multilinéaire, Hermann, Paris, 1948; these Rev. **10**, 231] in part B show how greatly his volume on the subject dominates the field. *I. Kaplansky.*

***Pickert, Günter. Normalformen von Matrizen.** Enzyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. I 1, 7. Band I. Algebra und Zahlentheorie. 1. Teil B. Algebra. Heft 3, Teil I, pp. 44–72. B. G. Teubner Verlagsgesellschaft, Leipzig, 1953. DM 7.50.

According to a footnote, this was intended to be part C of the preceding report, but was separated for editorial convenience.

The classical relations among matrices are equivalence, similarity, congruence, and orthogonal similarity. This report follows approximately this division of topics and in addition has a section on characteristic roots. The reviewer's impression on reading it is that few new ideas in the field have emerged in the past two decades. Authors have largely been engaged in perfecting old techniques. Normal matrices have attracted increasing attention and (of course) there is a steady trend toward studying these problems for matrices over rings which are as general as possible. The literature has been thoroughly combed and papers as late as 1953 are included. *I. Kaplansky.*

Ponting, F. W. A type of alternant. Proc. Edinburgh Math. Soc. (2) **9**, 20–27 (1953).

In the Schur function $\{\lambda\}=\{\alpha;(\lambda)\}$, where

$$\{\lambda\}\,|\alpha_s^{n-t}|=|\alpha_s^{l_t}|$$

and $l_t=\lambda_t+n-t$, we may replace α_j^k by

$$\alpha_j^{(k)}=(\alpha_j+\beta_1)(\alpha_j+\beta_2)\cdots(\alpha_j+\beta_k)$$

and consider the functions $\{\alpha;(\lambda);\beta\}$, where

$$\{\alpha;(\lambda);\beta\}\,|\alpha_s^{n-t}|=|\alpha_s^{(l_t)}|.$$

The connections between the Schur function $\{\alpha;(\lambda)\}$, the elementary symmetric function b and the complete homogeneous symmetric function H can now be generalized and the author proves that

$$\{\alpha;(\lambda);\beta\}=\sum\{\alpha;\lambda_1-r_1,\lambda_2-r_2,\cdots,\lambda_n-r_n\}b\{(l),(r)\},$$

where

$$b\{(l),(r)\}=|b(l_s,l_s-l_t+r_t)|=|H(2n-m_s,m_s-m_t+r_t')|.$$

The function $b\{(l),(r)\}$ can also be expanded as a polynomial in the β_i with positive integral coefficients; this last result is the hardest to prove and depends on the reduction of a product of Schur functions into a sum. Since the submission of this paper H. O. Foulkes [J. London Math. Soc. **25**, 268–275 (1950); these Rev. **12**, 793] has gone over the same ground in a somewhat different manner.

G. de B. Robinson (Toronto, Ont.).

Fadini, Angelo. Un'interpretazione mediante algebre dei campi finiti di Galois di ordine p^n. Rend. Accad. Sci. Fis. Mat. Napoli (4) **19** (1952), 42–44 (1953).

The author remarks that the Galois field with p^n elements may be regarded as an algebra over the prime field with p elements, with units $u_1,\cdots,u_n$ where $u_iu_j=u_{i+j-1}$, if $i+j-1<n$ and $u_iu_j=0$, if $i+j-1\geq n$. *W. Ledermann.*

Abstract Algebra

Balachandran, V. K. On disjunction lattices. J. Madras Univ. Sect. B. **23**, 15–21 (1953).

The author studies disjunction lattices, that is, distributive lattices L satisfying the property that given any two distinct elements a, b, of L there exists an element c such that one of ac, bc is 0 while the other is not. It is proved that a distributive lattice having an additive basis of minimals is a disjunction lattice, that a disjunction lattice is a Boolean algebra if and only if it is closed for product complements, and that in order that a lattice L closed for product-complements be a Boolean algebra either of the conditions (i) or (ii) is necessary and sufficient: (i) every principal ideal μ-ideal$\neq 1_\mu$ be the product of all minimal prime μ-ideals containing it; (ii) L is distributive and every principal α-ideal$\neq 1_\alpha$ is the product of all maximal α-ideals containing it. From this last result follows a theorem of Michiura [J. Osaka Inst. Sci. Tech. Part I. **1**, 129–133 (1949); these Rev. **12**, 387]: a complete lattice with completely distributive sums and products is a complete Boolean algebra if and only if every principal α-ideal is the product of all maximal α-ideals containing it. *H. Wallman* (Gothenburg).

Curzio, Mario. Alcune limitazioni sul minimo ordine dei reticoli modulari di lunghezza 3 contenenti sottoreticoli d'ordine dato. Ricerche Mat. **2**, 140–147 (1953).

It is shown that any modular lattice of length 3 and order exceeding $2(m^2-m+2)$ contains a sublattice of every order $n\leq 2m+4$. *G. Birkhoff* (Cambridge, Mass.).

Behrend, F. A. A system of independent axioms for magnitudes. J. Proc. Roy. Soc. New South Wales **87**, 27–30 (1953).

A set of independent axioms is given for positive "magnitudes" under $+$, $>$, following O. Hölder [Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math.-Phys. Cl. **53**, 1–64 (1901), pp. 13–14]. The axioms are trichotomy law for order, associative law for addition, solvability of $a+x=b$ if and only if $a<b$, and law of Archimedes. *G. Birkhoff.*

Dörge, Karl, und Schuff, Hans Konrad. Über Elimination in beliebigen Mengen mit allgemeinsten Operationen. Math. Nachr. **10**, 315–330 (1953).

Generalization of the results of a previous paper by K. Dörge [Math. Nach. **4**, 282–297 (1951); these Rev. **12**, 583] to the case of abstract algebras with multiple-valued operations not necessarily everywhere defined. *D. Zelinsky.*

Curtis, Charles W. Noncommutative extensions of Hilbert rings. Proc. Amer. Math. Soc. **4**, 945–955 (1953).

A ring S is said to be an extension of a subring R if R is in the center of S and if S has an identity element which lies in R. The extension is called finite if S is a finitely generated R-module and it is called integral if S satisfies the ascending chain condition for R-modules. Every integral extension is finite, and in case R is Noetherian, the converse is also true. The author shows that if S is an integral extension of R then the Jacobson radical of every homomorphic image of S coincides with the lower radical if and only if R is a Hilbert ring (i.e. the Jacobson radical of every homomorphic image of R is nil). He further shows that if S is an integral extension of R and if R contains a field K such that every field which is a homomorphic image of R is a finite algebraic extension of a field isomorphic with K, then every irreducible representation of S maps S onto a finite-dimensional simple algebra over K, and that the degrees of the absolutely irreducible representations of S are bounded. These results are applied to the study of the universal algebra A of an arbitrary finite-dimensional Lie algebra L over a field K of prime characteristic. By leaning on a recent paper of N. Jacobson [Amer. J. Math. **74**, 357–359 (1952); these Rev. **13**, 816], he is able to show that A is a finitely generated module over a Noetherian Hilbert ring R, which in turn is a homomorphic image of a finite polynomial extension $K[x_1,x_2,\cdots,x_n]$ over K. As a corollary it follows that A has "sufficiently many" finite-dimensional irreducible representations σ (that is, for each $s\neq 0$ there is some σ so that $s^\sigma\neq 0$). Another consequence is that L has a faithful completely reducible representation in a finite-dimensional space. Finally an example is presented (which the author

attributes to Jacobson) of a two-dimensional Lie algebra over a field with zero characteristic which has an infinite-dimensional irreducible representation. Some of the above results are shown to be valid also for the polynomial extension $S[x]$. *J. Levitzki* (Jerusalem).

Mauler, Heribert. Eine Darstellung für Identitäten zwischen den Kommutatoren eines Ringes. Math. Ann. 126, 410–417 (1953).

Let R be a ring with a unit element and let $R(x_1, \cdots, x_m)$ be the polynomial ring in $x_1, \cdots, x_m$ over R. If p_ν, p_ν' denote products of the x_μ and if the α_ν are elements of R, then the author defines an identity to exist between commutators if

$$\sum_{\nu=1}^{n} \alpha_\nu p_\nu (x_{i_\nu} x_{j_\nu} - x_{j_\nu} x_{i_\nu}) p_\nu' \equiv 0.$$

The principal result proved is: let $k_{\mu\nu} = x_\mu x_\nu - x_\nu x_\mu$; then every such identity between commutators is a linear combination with coefficients in R of the identities

1) $p(k_{\mu\nu} + k_{\nu\mu})p' \equiv 0$,

2) $p\{(x_\lambda k_{\mu\nu} + k_{\nu\mu} x_\lambda) + (x_\mu k_{\nu\lambda} + k_{\lambda\nu} x_\mu) + (x_\nu k_{\lambda\mu} + k_{\mu\lambda} x_\nu)\}p' \equiv 0$,

3) $p\{k_{\mu\nu} p' x_\rho x_\sigma + x_\nu x_\mu p' k_{\rho\sigma} + k_{\nu\mu} p' x_\sigma x_\rho + x_\mu x_\nu p' k_{\sigma\rho}\}p'' \equiv 0$.

The method of proof used is to identify a product of x's as a point and commutators of x's as a directed line segment in a line complex, and to then use results from here to obtain the algebraic result. As an application of his main result the author gives a proof of the theorem of Witt on the representation of a Lie ring as the Lie ring of an associative ring under the Lie product $a \circ b = ab - ba$. *I. N. Herstein.*

Amitsur, S. A. A general theory of radicals. II. Radicals in rings and bicategories. Amer. J. Math. 76, 100–125 (1954).

This is a continuation of the author's previous paper [same J. 74, 774–786 (1952); these Rev. 14, 347]. In this article he gives an axiomatic theory of radicals in non-associative rings. Let π be a property of rings, a π-ring is a ring which satisfies π, a π-ideal is a (two-sided) ideal which is a π-ring, a ring is π-semi-simple if it contains no nonzero π-ideals. A ring has a π-radical, $\pi(R)$, if there is a unique maximal π-ideal $\pi(R)$ such that $R/\pi(R)$ is π-semi-simple. Conditions on π to ensure the following are given: $\pi(R)$ exists for all rings R, $\pi(A) = \pi(R) \cap A$ for any ideal A in R, $\pi(R)$ contains all one-sided π-ideals, $\pi(R_f) = (\pi(R))_f$ where S_f is the ring of all row(column)-finite matrices with entries in S. For example, the author shows if π satisfies (1) 0 is a π-ring, (2) π is preserved by homomorphism, (3) if R is not a π-ring, R is homomorphic to a nonzero π-semi-simple ring, (4) every zero-ring is a π-ring, (5) every right ideal of a π-ring is a π-ring, (6) π-semi-simple rings contain no nonzero right π-ideals, then $\pi(R_f) = (\pi(R))_f$. The author points out that most of the work carries over to certain of the lattice-ordered bicategories of MacLane [Bull. Amer. Math. Soc. 56, 485–516 (1950); these Rev. 14, 133] and ends by applying his methods to rederive Brown and McCoy's theory of the F-radical [Trans. Amer. Math. Soc. 69, 302–311 (1950); these Rev. 12, 474].

There is some overlapping with a recent paper by Kuroš [Mat. Sbornik N.S. 33(75), 13–26 (1953); these Rev. 15, 194], but there only properties satisfying (1), (2), (3), are studied. *A. Rosenberg* (Evanston, Ill.).

Amitsur, S. A. A general theory of radicals. III. Applications. Amer. J. Math. 76, 126–136 (1954).

For part II see the preceding review. The author introduces a property of non-associative rings which yields a generalization of Baer's lower radical [same J. 65, 537–568 (1943); these Rev. 5, 88] and shows that it is the intersection of all the prime ideals of the ring. For two subsets A, B of a ring, AB is the set of elements ab; an ideal P is prime if, for any two ideals Q, S, $QS \subseteq P$ implies that either Q or S is in P. He further shows that solvable ideals in non-associative rings yield properties analogous to those of nilpotent ideals in the associative case. Restricting himself to associative rings he then studies Baer's lower radical, Levitzki's radical [Bull. Amer. Math. Soc. 49, 462–466 (1943); these Rev. 4, 238], the locally finite kernel also introduced by Levitzki [Trans. Amer. Math. Soc. 74, 384–409 (1953); these Rev. 14, 720] and shows that for each of these radicals $\pi(A) = \pi(R) \cap A$ and $\pi(R_f) = (\pi(R))_f$ (the notation is that of the preceding review.) Finally, the author enumerates various ring properties which yield radicals with most of the usual properties. The most interesting case is that of F.I. rings: A ring is an F.I. ring if in it and in each of its homomorphic images each non-nil right ideal contains a nonzero idempotent. *A. Rosenberg* (Evanston, Ill.).

Amitsur, A. Shimshon. Applications to a polynomial identity. Riveon Lematematika 7, 30–32 (1954). (Hebrew. English summary)

Consider the "standard polynomial"

$$S_n(x_1, \cdots, x_n) = \sum_{(i)} \pm x_{i_1} \cdots x_{i_n},$$

where the sum ranges over all permutations (i) of n letters and the sign is positive for even permutations and negative for odd permutations. Put

$$[xy] = xy - yx, \quad P_n(x, y) = S_n([xy], [xy^2], \cdots, [xy^n]),$$

and consider the identity $P_n(x, y) = 0$. The author shows that this identity, involving only two indeterminates, holds in every central simple algebra of order m^2 over an infinite field F if and only if $m \leq n$. Moreover, this identity holds for any algebraic algebra over F whose elements are of degrees bounded by n. This yields the following, almost immediate results. (1) A division ring is a central division algebra of order $\leq n^2$ if and only if every subdivision algebra generated by two elements is a central division algebra of order $\leq n^2$. (2) Every central division algebra of finite order is generated by two elements (Albert). (3) A sufficient condition that a, b generate a central division algebra of order n^2 is that $P_{n-1}(a, b) \neq 0$. (4) Every central division algebra A finite over its center C is a Kronecker product $A_0 \times C$, where A_0 is a division ring with two generators over the prime field. (5) Every division algebra whose elements are of degrees bounded by n is a central division algebra of order $\leq n^2$ (Jacobson). *J. Levitzki* (Jerusalem).

Morinaga, Kakutarō, and Nōno, Takayuki. On the linearization of a form of higher degree and its representation. J. Sci. Hiroshima Univ. Ser. A. 16, 13–41 (1952).

The authors first introduce a class of algebras which they call generalized Clifford algebras. They consider the direct product $P = G_1 \times G_2 \times \cdots \times G_n$ of n groups. With every element α of P, a basis element $e(\alpha)$ of the algebra A is associated and a suitable multiplication is defined. If each G_i is identified with the corresponding subgroup of P, then for $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in P$, $\alpha_i \in G_i$, one has

$\epsilon(\alpha)=\epsilon(\alpha_1)\epsilon(\alpha_2)\cdots\epsilon(\alpha_n)$. If β and γ are elements of the same G_i, then $\epsilon(\beta\gamma)=\epsilon(\beta)\epsilon(\gamma)$. For $\beta \in G_i$ and $\gamma \in G_j$ with $i>j$, one has formulas of the form $\epsilon(\beta)\epsilon(\gamma)=\rho_{\beta,\gamma}\epsilon(\gamma)\epsilon(\beta)$ where the $\rho_{\beta,\gamma}$ are elements of the underlying skew field K. In the second part of the paper, it is assumed that K is commutative and contains the mth roots of unity for a fixed positive integer m. It is shown that there exists a generalized Clifford algebra A in which the homogeneous form $x_1^m+x_2^m+\cdots+x_n^m$ of n indeterminates x_i becomes the product of m linear forms. An analogous result holds for $x_1^r+x_2^r+\cdots+x_n^r$ in the same algebra A, when r is a divisor of m such that r and m/r are relatively prime. Actually, the algebra A can be constructed directly from the group algebra Γ of a suitable group H. Here, H has a cyclic center $C=\{\omega\}$ of order m and H/C is a direct product of n cyclic groups of order m. The algebra A is obtained from Γ by identifying ω with a primitive mth root of unity in K. *R. Brauer.*

Kaplansky, Irving. Quadratic forms. J. Math. Soc. Japan **5**, 200–207 (1953).

Soit K un corps de caractéristique $\neq 2$ et qui ne soit pas réel au sens d'Artin-Schreier. Soient A l'indice du groupe multiplicatif $K^{(2)}$ des carrés des éléments non-nuls de K dans celui K^* des éléments non-nuls de K et soit C le plus petit entier n tel que toute forme quadratique de $n+1$ variables effectives et à coefficients dans K représente 0 d'une manière non-triviale. L'auteur émet l'hypothèse que $C \leqq A$ et la démontre dans les deux cas suivants: (1) $A \leqq 8$; (2) -1 est la somme d'au plus 4 carrés dans K. Dans une lettre personnelle, l'auteur a fait connaître aù référent que M. Kneser lui a signalé la démonstration simple suivante de son hypothèse: $f=\sum_{i=1}^{n}a_ix_i^2$ étant une forme quadratique dans K, soit f_r la somme de ses $r \leqq n$ premiers termes et soit M_r l'ensemble des valeurs de f_r quand les variables parcourent indépendemment K. Si $M_r=M_{r+1}$, on a, au sens de l'addition des sous-ensembles de K, $M_r+a_{r+1}K^{(2)}=M_r$, d'où, par induction, $M_r=M_r+a_r(K^{(2)}+K^{(2)}+\cdots+K^{(2)})$, quelque soit le nombre de termes $K^{(2)}$. Or K n'étant pas réel, tout $a \in K$ est une somme de carrés, donc appartient à la somme d'un certain nombre de $K^{(2)}$, d'où $M_r=M_r+a_{r+1}K=K$. Mais tout M_r est la réunion de $\{0\}$ et d'un certain nombre de classes (mod $K^{(2)}$) dans le groupe multiplicatif K^*, et, ainsi, le nombre des M_r distincts ne dépasse pas, puisque $M_r \subseteq M_{r+1}$, le nombre A de ces classes; ainsi, on a $M_r=K$ pour un $r \leqq A$, d'où résulte facilement $C \leqq A$. L'article contient encore le résultat suivant: si K n'admet, à l'isomorphie près, qu'une seule algèbre de quaternions, on a $C \leqq 4$. *M. Krasner.*

Kasch, Friedrich. Halblineare Abbildungen und die Rangrelation bei Galoisschen Erweiterungen. Arch. Math. **4**, 402–407 (1953).

Let A be a simple ring with minimum condition and let $E(A)$ be the endomorphism ring of the additive group of A. Then $E(A)$ contains the rings A^l and A^r of left and right multiplications by elements of A, anti-isomorphic and isomorphic to A respectively. An element $\mathfrak{v} \in E(A)$ is said to be A^r semi-linear if $a^r\mathfrak{v}=\mathfrak{v}(a\mathfrak{g})^r$, $a^r \in A^r$ where $\mathfrak{g}$ is an automorphism of A^r (and so of A). It follows easily that $\mathfrak{v}=\mathfrak{g}t^l$, $t^l \in A^l$. The author gives a simple, elegant proof of the following two statements. (1) The A^r semi-linear transformations $\mathfrak{v}_i=\mathfrak{g}_it_i^l$, $i=1, 2, \cdots, n$, are linearly dependent over A^r if and only if for some $\mathfrak{v}_1, \cdots, \mathfrak{v}_m$ (under a suitable re-numbering) the corresponding automorphisms $\mathfrak{g}_j=\mathfrak{g}_1a_j^l(a_j^{-1})^r$, $a_j \in A$, $j=1, \cdots, m$, and the elements $t_1a_1, \cdots, t_ma_m$ are linearly dependent over the center of A. (2) If R is a two-sided A^r-module which is generated over A^r by a finite number of A^r semi-linear transformations, then the same prevails for every two-sided A^r-submodule of R. As one corollary to this the author proves: If G is a semi-regular [for definition see Nakayama, Trans. Amer. Math. Soc. **73**, 276–292 (1952); these Rev. **14**, 240] automorphism group of a simple ring A and C is the fixed ring of G, then $(A:C)=(G:G_0)(T:Z)$ where T is the subring generated by those elements of A which induce inner automorphisms of A which are contained in G, G_0 is this set of inner automorphisms and Z is the center of A. *I. N. Herstein.*

Dieudonné, Jean. On semi-simple Lie algebras. Proc. Amer. Math. Soc. **4**, 931–932 (1953).

By a completely elementary argument of half a page, the following result is obtained. Let $\mathfrak{g}$ be a finite-dimensional Lie Algebra over an arbitrary field. Suppose that $\mathfrak{g}$ has no non-zero abelian ideal, and that there is a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$. Then $\mathfrak{g}$ is a direct sum of simple (non-abelian) Lie algebras. With the easy observation that the non-degeneracy of the Killing form of a Lie algebra $\mathfrak{g}$ implies that $\mathfrak{g}$ has no non-zero abelian ideals, this gives the sufficiency of E. Cartan's criterion for semi-simplicity of a Lie algebra. *G. Hochschild.*

Murakami, Shingo. Supplements and corrections to my paper: On the automorphisms of a real semi-simple Lie algebra. J. Math. Soc. Japan **5**, 105–112 (1953).

The supplements consist in a simplification of the argument of the paper cited in the title [same J. **4**, 103–133 (1952); these Rev. **14**, 531]. Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, $\mathfrak{g}_u$ a compact form of $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ which contains a Cartan subalgebra $\mathfrak{h}_u$ of $\mathfrak{g}_u$ and S an involutive automorphism of $\mathfrak{g}$. It is proved directly and quite simply that S is conjugate to (i.e. transformable by an inner automorphism into) an operation S' which leaves $\mathfrak{h}$ invariant and which induces on $(-1)^{\frac{1}{2}}\mathfrak{h}_u$ a "particular rotation" ρ, i.e. a rotation which permutes among themselves the roots of some fundamental system. Moreover, S' may be written in the form $S_0 \exp(\mathrm{ad}\,\lambda)$, where S_0 is an automorphism which may be determined explicitly in terms of ρ, while λ is an element of $\mathfrak{h}_u$ fixed under ρ.

The corrections are as follows: (a) the proof of lemma 12 of the previous paper is admitted to be incomplete; but this is immaterial, since said lemma is superseded by the results of the present paper; (b) the general statements about the automorphisms of algebras of type A_n do not apply to the trivial case where $n=1$; (c) a few misprints.

C. Chevalley (Nagoya).

Kirchhoff, A. Beiträge zur topologischen linearen Algebra. Compositio Math. **11**, 1–36 (1953).

Let V be a finite-dimensional vector space over the field of real numbers. Two endomorphisms A, A' of V are called positively similar if there exists an automorphism T of V of determinant >0 such that $A'=TAT^{-1}$. A similarity class may either remain a class of positive similarity, in which case it is (and its elements are) called two-sided, or split into two classes of positive similarity, in which case it is (and its elements are) called one-sided. An endomorphism A is proved to be two-sided if and only if V is the direct sum of two A-invariant subspaces whose dimensions are both odd. Any class of positive similarity in the set of all endomorphisms of V is connected; a one-sided class of similarity has two connected components. To every one-sided class of positive equivalence there is intrinsically

associated an orientation of V, and the orientations attached to the two classes of positive similarity which make up a class of similarity are distinct. The orientation attached to a one-sided endomorphism A is a discontinuous function of A if the dimension n of V is ≥ 4, for, in that case, the set of all one-sided endomorphisms is connected; if $n=2$, then the orientation depends continuously on A.

The author calls I-mapping an endomorphism which has no invariant space of odd dimension. Let $\mathfrak{J}$ be the set of these mappings; it is dense in the set of one-sided endomorphisms and open in the set of all endomorphisms. Let $\mathfrak{J}'$ be the set of the mappings A in $\mathfrak{J}$ with $A^2=-E$ (E=the identity); V being provided with a euclidean metric, let $\mathfrak{J}''$ be the set of elements in $\mathfrak{J}$ which are orthogonal. Then $\mathfrak{J}'$ is a deformation retract of $\mathfrak{J}$ and $\mathfrak{J}''$ of $\mathfrak{J}'$. The set $\mathfrak{J}'$ is a one-sided class of similarity; thus, each of $\mathfrak{J}$, $\mathfrak{J}'$, $\mathfrak{J}''$ splits into two connected components. If V is of dimension n, then each component of $\mathfrak{J}''$ is fibered over the $(n-2)$-sphere, the fiber being homeomorphic to a component of the space $\mathfrak{J}''$ relative to a space of dimension $n-2$. The two components in $\mathfrak{J}''$ may be separated by the sign of the pfaffian of the matrix which represents an element of $\mathfrak{J}''$ with respect to an orthogonal base of the space.

A manifold M is called an I-manifold if there exists a continuous family (A_x), $x \in M$, where A_x is an I-automorphism of the tangent space to M at x. Such a manifold is almost-complex (and, of course, conversely). If the n-sphere is an I-manifold, then the $(n+1)$-sphere admits an absolute parallelism; thus, if n is not of the form 2^k-2, the n-sphere is not an I-manifold. *C. Chevalley* (Nagoya).

Aubert, Karl Egil. Über Bewertungen mit halbgeordneter Wertgruppe. Math. Ann. **127**, 8–14 (1954).

Define an r-system of r-ideals in a partially ordered multiplicative group Γ following Prüfer and Lorenzen [cf. Lorenzen, Math. Z. **45**, 533–553 (1939); these Rev. **1**, 101]. An r-valuation of a commutative field K into Γ is a multiplicative homomorphism w of K^* into Γ with $w(a-b)$ in the r-ideal generated by $w(a)$ and $w(b)$ for every a, b of K^*. This valuation concept includes as special cases (1) Krull's generalized valuations, obtained by specializing the r-system to be the largest possible one: $\mathfrak{a}_x=\bigcup_{a \in \mathfrak{a}} (a)$; and (2) the semivaluations of the reviewer [Bull. Amer. Math. Soc. **54**, 1145–1150 (1948); these Rev. **10**, 426] obtained by taking the smallest r-system: $\mathfrak{a}_r=\bigcap_{(a) \supset \mathfrak{a}} (a)$. An r-valuation naturally defines an r-valuation ring and an r-system of r-ideals in it. The r-valuation rings may be characterized by the property that each r-ideal is an ideal. *D. Zelinsky.*

Theory of Groups

***Kuroš, A. G. Teoriya grupp. [The theory of groups.] 2d ed.** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 467 pp. 16.85 rubles.

The manuscript of the first edition [these Rev. **9**, 267] was completed in 1940. In the ensuing decade group theory changed so much that the author felt that a mere revision was inadequate; this is in large part a new book.

The new chapter headings are as follows. I. Definition of a group. II. Subgroups. III. Normal subgroups. IV. Endomorphisms and automorphisms, groups with operators. V. Series of subgroups, direct products, defining relations. VI. Foundations of the theory of abelian groups. VII. Primary and mixed abelian groups. VIII. Torsion-free abelian groups. IX. Free products and free groups. X. Finitely generated groups. XI. Direct decompositions, lattices. XII. Group extensions. XIII. Finiteness conditions, Sylow subgroups, related problems. XIV. Solvable groups. XV. Nilpotent groups.

New material occurs principally from the seventh chapter on. In Chapter 7 Kulikoff's characterization of direct sums of cyclic groups [Mat. Sbornik N.S. **16(58)**, 129–162 (1945); these Rev. **8**, 252] is proved at once, and is then applied to the older special cases (groups of bounded order and countable groups with no elements of infinite height). Kulikoff's theory of basic subgroups is also included. In Chapter 8 the author's own theory of torsion-free groups of finite rank is omitted in favor of the theory of direct sums of groups of rank one, due largely to Baer [Duke Math. J. **3**, 68–122 (1937)]. The 38 pages of Chapter 10 include accounts of recent work of B. H. Neumann, H. Neumann, M. Hall, and Malcev. Chapter 10 contains a complete proof of Gruschko's theorem [Mat. Sbornik N.S. **8(50)**, 169–182 (1940); these Rev. **2**, 215], as well as the theorem of G. Higman and the Neumanns [J. London Math. Soc. **24**, 247–254 (1949); these Rev. **11**, 322] asserting that any countable group can be embedded in a group with two generators. Chapter 11 uses lattice-theoretic methods (as in recent papers by Baer, Kuroš and others) to discuss the existence of isomorphic refinements of direct decompositions of a group. Chapter 12 includes an introduction to the Eilenberg-MacLane homology theory of abstract groups. The final three chapters follow the lines laid down by Kuroš and Cernikov [Uspehi Matem. Nauk (N.S.) **2**, no. 3(19), 18–59 (1947) = Amer. Math. Soc. Translation no. **80** (1953); these Rev. **10**, 677; **14**, 618]. There is an up-to-date bibliography occupying 21 pages.

At the end of the preface the author expresses the hope that his book will be supplemented by two further ones: on finite groups and on groups of transformations.

I. Kaplansky (Chicago, Ill.).

Vagner, V. V. The theory of generalized heaps and generalized groups. Mat. Sbornik N.S. **32(74)**, 545–632 (1953). (Russian)

Dans une note récente [Doklady Akad. Nauk SSSR (N.S.) **84**, 1119–1122 (1952); ces Rev. **14**, 12] l'auteur avait introduit le concept de groupe généralisé (ensemble muni d'une multiplication associative pour laquelle tous les idempotents sont permutables et où tout élément x admet un "inverse généralisé" x^{-1} tel que $xx^{-1}x=x$) et avait montré qu'il permettait de caractériser abstraitement les ensembles multiplicatifs de relations biunivoques sur un même ensemble qui contiennent la symétrique de toute relation leur appartenant. Le présent travail consiste essentiellement à construire le concept d'amas généralisé afin de permettre une caractérisation analogue des ensembles de relations biunivoques sur deux ensembles qui contiennent $\Sigma_1\Sigma_2^{-1}\Sigma_3$ pour tout triple Σ_1, Σ_2, Σ_3 d'éléments leur appartenant et a le relier au concept de groupe généralisé.

Etant donné deux ensembles A et B, si à trois relations binaires $R_1 \subset A \times B$, $R_2 \subset A \times B$, $R_3 \subset A \times B$ on associe $[R_1, R_2, R_3]=R_1R_2^{-1}R_3$, on définit ainsi une opération ternaire sur $\mathfrak{P}(A \times B)$ qui le transforme en demi amas dans le sens suivant: Un ensemble muni d'une opération ternaire $[\ ,\ ]$ sera appelé demi amas lorsque pour tout quintuple

k_1, k_2, k_3, k_4, k_5 d'éléments de cet ensemble on a

$$[[k_1, k_2, k_3], k_4, k_5] = [k_1, [k_4, k_3, k_2], k_5] = [k_1, k_2, [k_3, k_4, k_5]].$$

Un élément k sera dit biunitaire lorsque, quel que soit k, $[k, h, h] = [h, h, k] = k$. Par définition un amas est un demi amas dont tous les éléments sont biunitaires. L'opération ternaire $[g_1, g_2, g_3] = g_1 g_2^{-1} g_3$ qui permet de considérer tout groupe comme un amas avait été étudiée par H. Prüfer [Math. Z. 20, 165–187 (1924)] pour les groupes abéliens et par R. Baer [J. Reine Angew. Math. 160, 199–207 (1929)] et J. Certaine [Bull. Amer. Math. Soc. 49, 869–877 (1943); ces Rev. 5, 227] pour les groupes.

Le travail comprend cinq parties: Dans la première on étudie idéaux et équivalences dans les demi amas et on donne une généralisation d'un théorème du à Lyapin [Izvestiya Akad. Nauk SSSR. Ser. Mat. 14, 179–192 (1950); ces Rev. 11, 575]. Dans la seconde on établit la liaison entre demi amas et demi groupes involutifs (c'est-à-dire, ensembles munis d'une multiplication associative et d'un antiautomorphisme d'ordre 2 pour cette multiplication). Tout demi groupe involutif est de manière évidente un demi amas. Réciproquement, si K est un demi amas ayant un élément biunitaire e et si l'on pose $k_1 k_2 = [k_1, e, k_2]$, $k^{-1} = [e, k, e]$, K devient un demi groupe involutif K_e et tous les demi groupes involutifs K_x sont isomorphes, x parcourant l'ensemble des éléments biunitaires. Tout demi amas peut être plongé dans un demi groupe involutif. Dans la troisième on définit le concept d'amas généralisé: c'est un demi amas dont tous les éléments sont idempotents (c'est-à-dire, $x = [x, x, x]$) et bipermutables (c'est-à-dire, pour tout triple k, k_1, k_2 on a $[k, k_1, k_1, k_2, k_2] = [k, k_2, k_2, k_1, k_1]$, $[k_1, k_1, k_2, k_2, k] = [k_2, k_2, k_1, k_1, k]$). Tout group généralisé considéré comme demi amas est un amas généralisé. Tout amas généralisé peut être plongé dans un groupe généralisé.

La quatrième partie définit et étudie deux types spéciaux de relations binaires dans les amas et les groupes généralisés: les relations de communion et d'ordre canonique. On y trouve une généralisation d'un théorème de D. Rees [J. London Math. Soc. 22, 281–284 (1948); ces Rev. 9, 568] relative à l'ensemble des équivalences R sur un groupe généralisé G telles que G/R est un groupe. Dans la cinquième partie on montre que, pour que K soit un amas généralisé, il faut et il suffit qu'il soit isomorphe à un demi amas de relations biunivoques entre éléments de deux ensembles. (L'ordre canonique est alors l'ordre d'inclusion et la relation de communion entre Σ_1 et Σ_2 a lieu lorsque $\Sigma_1 \cup \Sigma_2$ est une biunivoque du demi amas.) On donne un certain nombre de propriétés des amas généralisés de relations quasi fonctionnelles R (c'est-à-dire, telles que $RR^{-1} \subset \Delta$).

L'exposé fait systématiquement appel à la théorie des relations binaires en s'appuyant sur le travail du rapporteur [Bull. Soc. Math. France 76, 114–155 (1948); ces Rev. 10, 502]. Il est signalé dans la préface que les groupes et amas généralisés de biunivoques jouent un rôle important en géométrie, par exemple, pour les systèmes de coordonnées locaux [Vagner, Doklady Akad. Nauk SSSR (N.S.) 81, 981–984 (1951); ces Rev. 13, 688]. Encore a titre d'exemple, les transformations conformes du plan définies par des fonctions analytiques a une seule branche pour lesquelles l'équation différentielle de définition exprime la condition de Cauchy-Riemann pour les fonctions analytiques forment un groupe généralisé.

J. Riguet (Paris).

Liber, A. E. On symmetric generalized groups. Mat. Sbornik N.S. 33(75), 531–544 (1953). (Russian)

A étant un ensemble donné, on désigne par S_A et on appelle groupe généralisé symétrique le groupe généralisé au sens de Vagner [cf. l'analyse ci-dessus] de toutes les relations biunivoques entre éléments de A. Le présent travail résout pour S_A les problèmes résolus par Malcev [Mat. Sbornik N.S. 31(73), 136–151 (1952); ces Rev. 14, 349] pour le demi groupe de toutes les applications de A dans A. Il se divise en trois parties: Dans la première, on donne une caractérisation algébrique des groupes généralisés symétriques basée sur le concept d'élément "primitif". Cette caractérisation est, du reste, différente de celle donnée indépendamment par E. S. Lyapin [Doklady Akad. Nauk SSSR (N.S.) 88, 13–15 (1953); ces Rev. 15, 395]. On y trouve une démonstration nouvelle du théorème de Schreier: tout automorphisme d'un groupe généralisé symétrique est intérieur.

Dans la seconde partie on montre que l'ensemble des idéaux de S_A coïncide avec l'ensemble des idéaux J_ξ quand ξ parcourt les nombres cardinaux, J_ξ étant constitué par l'ensemble des biunivoques $\Sigma \subset A \times A$ dont la puissance $\#\Sigma$ est inférieure à ξ. On étudie certaines équivalences compatibles: On désigne par $\bar{\delta}_\xi$ l'équivalence compatible $(J_\xi \times J_\xi) \cup \Delta$. Si n est entier fini, on désigne par δ_n l'équivalence compatible définie par: $\Sigma_1 \equiv \Sigma_2 \bmod \delta_n$ si et seulement si $\Sigma_1 = \Sigma_2$ ou $\#\Sigma_1 < n$ et $\#\Sigma_2 < n$ ou $\#\Sigma_1 = \#\Sigma_2 = n$, $\mathrm{pr}_1\Sigma_1 = \mathrm{pr}_1\Sigma_2$, $\mathrm{pr}_2\Sigma_1 = \mathrm{pr}_2\Sigma_2$ et $\Sigma_2^{-1}\Sigma_1 \in \mathfrak{N}_n$ où $\mathfrak{N}_n$ est un sous groupe distingué du groupe symétrique de degré n. Si ξ est un cardinal infini, on désigne par δ_ξ l'équivalence compatible définie par $\Sigma_1 \equiv \Sigma_2 \bmod \delta_\xi$ si et seulement si

$$\max\{\#\Sigma_1 \cap \Sigma_2', \#\Sigma_2 \cap \Sigma_1'\} < \xi$$

(Σ' désignant le complémentaire de Σ).

Dans la troisième partie, on montre que toute relation d'équivalence compatible avec la structure de S_A est intersection d'équivalences de la forme $\bar{\delta}_\rho \delta_\xi$ $(= \bar{\delta}_\rho \cup \delta_\xi)$ où $1 \leqq \xi \leqq \rho \leqq \eta + 1$, η désignant la puissance de A.

J. Riguet (Paris).

Asano, Keizo, and Murata, Kentaro. Arithmetical ideal theory in semigroups. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. 4, 9–33 (1953).

This paper extends to semigroups previous work by Asano on arithmetical ideal theory in non-commutative rings [Osaka Math. J. 1, 98–134 (1949); these Rev. 11, 154]. A similar extension of the work of Henke [Abh. Math. Sem. Hamburg. Univ. 11, 311–332 (1935)] had been given by Kawada and Kondô [Jap. J. Math. 16, 37–45 (1939); these Rev. 1, 164]. While the latter paper develops the theory of v-ideals (in a non-commutative semigroup S), the present paper uses an idea due to Lorenzen [Math. Z. 45, 533–553 (1939); these Rev. 1, 101] of developing the theory of an arbitrary system of ideals in S. In this way the theory includes that of ordinary ideals in a ring, as well as that of v-ideals.

Let S be a semigroup with identity element 1, and let S^* be the group of units of S. [As far as the reviewer can see, all the results of this paper hold without any modification if we let S^* be any fixed subgroup of the group of units of S. If so, they include those of Kawada and Kondô, who take S^* to be an arbitrary fixed subgroup of the center of S.] A subsemigroup $\mathfrak{o}$ of S containing 1 is called an "order" of S if every element of S is expressible in both the forms $\alpha^{-1}a$ and $b\beta^{-1}$, with $a, b \in \mathfrak{o}$ and $\alpha, \beta \in \mathfrak{o} \cap S^*$. A subset $\mathfrak{a}$ of S is called a "left (right) $\mathfrak{o}$-ideal" if (1) $\mathfrak{o}\mathfrak{a} \subseteq \mathfrak{a}$ ($\mathfrak{a}\mathfrak{o} \subseteq \mathfrak{a}$), (2)

$\mathfrak{a} \cap S^*$ is not vacuous, and (3) there exists $\lambda \in S^*$ such that $\mathfrak{a}\lambda \subseteq \mathfrak{o}$ ($\lambda\mathfrak{a} \subseteq \mathfrak{o}$). Let L be the set of all two-sided $\mathfrak{o}$-ideals of S. Let there be given a mapping $\mathfrak{a} \to \bar{\mathfrak{a}}$ of L into itself satisfying: (1) $\mathfrak{a} \subseteq \bar{\mathfrak{a}}$, (2) $\bar{\bar{\mathfrak{a}}} = \bar{\mathfrak{a}}$, (3) $\mathfrak{a} \subseteq \mathfrak{b}$ implies $\bar{\mathfrak{a}} \subseteq \bar{\mathfrak{b}}$, (4) $\overline{\mathfrak{a}\mathfrak{b}} \subseteq \overline{\mathfrak{a}\mathfrak{b}}$, where $\mathfrak{a}\mathfrak{b}$ means the set of all ab with $a \in \mathfrak{a}$, $b \in \mathfrak{b}$. Here $\mathfrak{a}$ is called a closed $\mathfrak{o}$-ideal if $\bar{\mathfrak{a}} = \mathfrak{a}$. (For example, if S is a ring and $\bar{\mathfrak{a}}$ is the set of all finite sums of elements of $\mathfrak{a}$, then the closed $\mathfrak{o}$-ideals are just the usual $\mathfrak{o}$-ideals of S.) An analogous definition of an ideal closure operator is given in the set G of one-sided ideals associated with a complete set of equivalent maximal orders of S. All the theory of two-sided ideals in the paper of Asano cited above, except where reference is made to residue class rings, holds almost verbatim. An abstract formulation is first given for a residuated lattice, and then applied to L. The same procedure is followed for the theory of one-sided ideals, first giving structure and decomposition theorems in a "lattice-ordered (Brandt) gruppoid", and then applying the results to G.

A. H. Clifford (Baltimore, Md.).

Carlitz, L. A note on abelian groups. Proc. Amer. Math. Soc. **4**, 937–938 (1953).

Let $a_1, a_2, \cdots, a_h$ denote the elements of a finite abelian group A. Then A is said to have the property M if it is possible to find a permutation $b_1, b_2, \cdots, b_h$ of the a_i such that $a_1b_1, a_2b_2, \cdots, a_hb_h$ are distinct. It is proved that an abelian group A possesses the property M unless it contains exactly one element of order 2. As the author points out, this has already been proved by L. J. Paige [Bull. Amer. Math. Soc. **53**, 590–593 (1947); these Rev. **9**, 6].

R. A. Rankin (Birmingham).

de Bruijn, N. G. On the factorization of cyclic groups. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 370–377 (1953).

The author extends the results obtained by him in an earlier paper [same Proc. **56** = Indagationes Math. **15**, 258–264 (1953); these Rev. **15**, 8]. In the terminology used there, he shows that all cyclic groups of order $p^\lambda q$ (p, q prime, $\lambda \geqq 1$) are 'good'. This appears as a particular case of a more general theorem concerning the divisibility of polynomials $A(x)$ and $B(x)$ with integral non-negative coefficients such that

$$A(x)B(x) \equiv T(x) \pmod{x^n - 1} \quad (n = p^\lambda q),$$

where $T(x)$ is a polynomial of the same type; in the application $T(x)$ is the polynomial $1+x+x^2+\cdots+x^{n-1}$. It is also shown that if a cyclic group G is of order n, where n has not more than two different prime factors, then the existence of a factorisation $G = AB$, in which A has a prime number of elements, implies that either A or B is periodic.

Further, if for two polynomials $A(x)$, $B(x)$ with integral coefficients,

$$A(x)B(x) \equiv 1+x+\cdots+x^{n-1} \pmod{x^n - 1},$$

then, in certain circumstances, one of the factors, say $B(x)$, can be written as $B(x) = B_0(x)+B_1(x)+\cdots+B_{p-1}(x)$ (p a prime divisor of n), where

$$A(x)B_j(x) \equiv x^{jn/p}A(x)B_0(x) \pmod{x^n - 1} \quad (0 \leqq j < p),$$

and the polynomials $B_j(x)$ have integral, though not necessarily non-negative coefficients. This result is of interest in view of a conjecture of Hajós regarding the 'quasiperiodic' nature of every factorisation $G = AB$. This result uses a theorem of Rédei [Acta. Math. Acad. Sci. Hungar. **1**, 197–207 (1950); these Rev. **13**, 623] of which the author gives a new and complete proof.

Throughout the paper the group-theoretical problems considered are treated as particular cases of the associated problems concerning polynomial ideals. The author also gives an example of a bad cyclic group of order 72.

R. A. Rankin (Birmingham).

Greco, Donato. Su alcuni gruppi finiti che sono somma di cinque sottogruppi. Rend. Sem. Mat. Univ. Padova **22**, 313–333 (1953).

The author proves that a finite group G is the sum of five proper subgroups $G_1, \cdots, G_5$, any two of which intersect in a fixed subgroup H, if and only if there exists a normal subgroup N of G, $N \subseteq H$, such that G/N has one of the following structures: (A) abelian of type (2, 2, 2) or dihedral of order 8; (B) alternating group of degree 4; (C′) abelian of type (2, 2, 2, 2); (C″) the group obtained from the abelian group $\{a_1\} \times \{a_2\} \times \{a_3\} \times \{a_4\}$ of type (2, 2, 2, 2) by adjoining the automorphism b defined by $a_1^b = a_2$, $a_2^b = a_1a_3$, $a_3^b = a_4$, $a_4^b = a_3a_4$. The proof proceeds by an examination of cases, according to the possible values of the indices $G:G_i$ and $G:H$, corresponding to (A), (B), (C) above.

D. G. Higman (Montreal, Que.).

Huppert, Bertram. Monomiale Darstellung endlicher Gruppen. Nagoya Math. J. **6**, 93–94 (1953).

A finite group G is said to be an M-group if every irreducible representation of G is similar to a monomial representation. A group G is said to be oversolvable (überauflösbar) if it has a chief series whose factor groups all have prime order. The author proves the following theorem which is an extension of earlier results of Zassenhaus [Abh. Math. Sem. Hamburg. Univ. **11**, 187–220 (1936)] and Itô [Nagoya Math. J. **4**, 79–81 (1952); these Rev. **13**, 909]: Suppose that a finite group G has a solvable normal subgroup N with all Sylow subgroups abelian and that G/N is oversolvable, then G is an M-group. He also proves that if G/M and G/N are oversolvable, then so is $G/(M \cap N)$. *R. M. Thrall.*

Lazard, Michel. Problèmes d'extension concernant les N-groupes; inversion de la formule de Hausdorff. C. R. Acad. Sci. Paris **237**, 1377–1379 (1953).

A statement of results about groups

$$G = H_1 \supset H_2 \supset \cdots \supset H_i \supset \cdots$$

with a descending central series (H_i) which is separating, in the sense that $\bigcap_i H_i = (1)$. For example, if G is complete with respect to this series and if, for all i, H_i/H_{i+1} satisfies the conditions (i) that none of its elements have orders $2, 3, \cdots, i$, (ii) each of its elements has an $i!$-th root in H_i/H_{i+1}, then given $x, y \in G$ there exists a uniquely determined sequence (a_i) such that $a_i \in H_i$ and, for any integer t, $x^t y^t = a_1^t a_2^{t^2} \cdots a_i^{t^i} \cdots$. Putting $x+y = a_1$, $[x, y] = a_2^2$, one turns G into a Lie ring and can recover the group structure from this by the Hausdorff formula. *P. Hall.*

Baeva, N. V. Completely factorizable groups. Doklady Akad. Nauk SSSR (N.S.) **92**, 877–880 (1953). (Russian)

A group is said to be completely factorizable if each subgroup A of G has a complement D in G; i.e., there exists a subgroup D such that $AD = G$ and $A \cap D = 1$. P. Hall [J. London Math. Soc. **12**, 201–204 (1937)] showed that finite completely factorizable groups are those which are subgroups of direct products of a finite number of groups of square-free order. Without proofs, the author states that, even in the infinite case, generalizing a result of Hall, (1) the commutator subgroup of a completely complemented

group is abelian; (2) G is locally abelian; (3) if G is a p-group it is elementary abelian; (4) G is, equivalently, the semi-direct product of a normal abelian subgroup A and of an abelian subgroup D, where both A and D split into direct products of cyclic groups of prime orders which, in the case of A, are normal subgroups of G. Further typical results are: (5) a locally normal group is completely factorizable if and only if it is a subgroup of a (possibly infinite) direct product of the type given by Hall, above. (6) Each group G possessing an ascending normal series with cyclic factors with non-repeating prime orders is completely factorizable. [See the following review.] *F. Haimo.*

Černikov, S. N. Groups with systems of complementary subgroups. Doklady Akad. Nauk SSSR (N.S.) **92**, 891–894 (1953). (Russian)

[See the preceding review.] P. Hall [J. London Math. Soc. **12**, 198–200 (1937)] proved that a finite group G is solvable if and only if every Sylow subgroup of G is complemented. Results of a similar nature for (possibly infinite) groups are stated here without proof. For instance, each serving subgroup S ($nx \in S$ implies the existence of $y \in S$ with $nx=ny$) of pericdic abelian G is complemented if and only if G is the direct product of a complete group G^* ($nx \in G^*$ implies $x \in G^*$) and of primary groups each of which has a bound on the order of its elements. If a group G has an increasing normal series with complete abelian factors, and if in G each complete abelian subgroup is complemented, then G is a complete abelian group. A locally nilpotent group in which each normal subgroup is complemented is the direct product of elementary abelian groups. A group G with complemented normal subgroups and with a principal ascending series with cyclic factors of prime order is completely factorizable, and conversely. (Cf. the last result mentioned in the preceding review.)

F. Haimo (St. Louis. Mo.).

Krull, Wolfgang. Über gewisse Homomorphismen von Polynomgruppen. Math. Ann. **126**, 377–380 (1953).

Let G be a group, F a free group, and X their free product. Let $x \in X$ have images g and f in the natural homomorphisms of X on G and F. Then if g belongs to the centre of G and $f \neq 1$, there is a normal subgroup of X containing x and intersecting G in 1. The author conjectures that the assumption that g belongs to the centre may be unnecessary; the assumption $f \neq 1$ is necessary. *Graham Higman.*

Schiek, Helmut. Bemerkung über eine Relation in freien Gruppen. Math. Ann. **126**, 375–376 (1953).

A short and direct proof of the proposition: If $w(\neq 1)$, $s_1, \cdots, s_n$ are elements of a free group, and $m_1, \cdots, m_n$ integers such that

$$s_1^{-1}w^{m_1}s_1s_2^{-1}w^{m_2}s_2\cdots s_n^{-1}w^{m_n}s_n=1,$$

then $m_1+m_2+\cdots+m_n=0$. The proposition is, of course, true in any group in which the intersection of the terms of the lower central series is 1, and the factor groups of its successive terms are torsion-free. *Graham Higman.*

Fuchs, L. Über die Zerlegung einer Gruppe nach zwei Untergruppen. Monatsh. Math. **57**, 109–112 (1953).

Let H, K be subgroups of the group G, and let A be the intersection for all x in G of the complexes $x^{-1}HxK$. If A^{-1} denotes the inverse complex, then $[H; K]=A \cap A^{-1}$ is a subgroup of G. The author gives two results in terms of this construction. First, the double coset decomposition of G with respect to the modulus (H, L) is a refinement of that with respect to (H, K) if and only if $L \leqq [H; K]$, or equivalently $[H; L] \leqq [H; K]$. Secondly, the complexes of the double coset decomposition of G with respect to (H, K) form a group under complex multiplication if and only if $[H; K]=[K; H]$ is a normal subgroup of G.

D. G. Higman (Montreal, Que.).

Kuga, Michio. On two theorems of Siegel. Sci. Papers Coll. Gen. Ed. Univ. Tokyo **3**, 1–11 (1953).

Let $\mathfrak{G}$ be the real special linear group of degree n and Γ its discrete subgroup consisting of all matrices in $\mathfrak{G}$ which have integral elements. The homogeneous space $\Gamma\backslash\mathfrak{G}$ formed by the right cosets Γ_g of Γ in $\mathfrak{G}$ has a $\mathfrak{G}$-invariant measure (dx). Let V be the $\frac{1}{2}n(n+1)$-dimensional vector space formed by all real symmetric matrices of degree n and Λ the lattice point set in V formed by all matrices in V which have integral elements. The set of all non-singular matrices in Λ is denoted by Λ_1. The image $\Lambda[\mathfrak{X}]=\mathfrak{X}'\Lambda\mathfrak{X}$ of Λ in the mapping $\mathfrak{S} \to \mathfrak{X}'\mathfrak{S}\mathfrak{X}$ (where $\mathfrak{X} \in \mathfrak{G}$) of Λ into itself is again a lattice point set in V. Now if f is a real-valued continuous function on V, which is $\neq 0$ on some compact set only ("à support compact"), the author considers the integral of $\sum f(\mathfrak{X}'\mathfrak{S}\mathfrak{X})$ (the summation to be extended over all $\mathfrak{S} \in \Lambda_1$) over the homogeneous space $\Gamma\backslash\mathfrak{G}$ and proves that it is convergent if $n \geqq 3$. The analogous theorem for skew-symmetric instead of symmetric matrices is true without exception. *H. D. Kloosterman* (Leiden).

Gamba, Augusto, e Radicati, Luigi A. Sopra un teorema per la riduzione di talune rappresentazioni del gruppo simmetrico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 632–634 (1953).

Let $(u_1, u_2, \cdots)$ be any (finite) sequence of integers. We define $D(u_1, u_2, \cdots)$ to be $(u_1, u_2, \cdots)$ if $u_1 \geqq u_2 \geqq \cdots$ and to be the null sequence otherwise. We define $\Delta_j^-(u_1, u_2, \cdots)$ to be $(u_1, \cdots, u_j-1, \cdots)$ and define $\Delta_j^+(u_1, u_2, \cdots)$ to be $(u_1, \cdots, u_j+1, \cdots)$. Let $A^-=\Delta_1^-+\Delta_2^-+\cdots$, and let $A^+=\Delta_1^++\Delta_2^++\cdots$. If $u_1 \geqq u_2 \geqq \cdots$ is a partition of n and if $\chi(u_1, u_2, \cdots; \alpha_1, \alpha_2, \cdots)$ is the irreducible character of the symmetric group S_n for the class with α_1 1-cycles, α_2 2-cycles, $\cdots$, and if χ applied to the null class is defined to be zero, the authors show that

$$(DA^+DA^- - I)\chi(u_1, u_2, \cdots, \alpha_1, \alpha_2, \cdots) = \chi(n-1, 1; \alpha_1, \alpha_2, \cdots)\chi(u_1, u_2, \cdots, \alpha_1, \alpha_2, \cdots).$$

R. M. Thrall (Ann Arbor, Mich.).

Bruhat, François. Irréductibilité des représentations induites des groupes de Lie. C. R. Acad. Sci. Paris **238**, 38–40 (1954).

The author continues his study of induced representations of Lie groups begun in a previous note [same C. R. **237**, 1478–1480 (1953); these Rev. **15**, 398]. Let G be a Lie group and Γ a closed subgroup. The author assumes throughout that the number of double cosets of Γ in G is finite. Under these assumptions he outlines his results on the relationship of the intertwining operators between U^a and U^b to certain distributions (in the sense of L. Schwartz) on G; here U^a and U^b are the induced representations generated by one-dimensional representations a, b of Γ. *F. I. Mautner.*

Bruhat, François. Représentations induites des groupes de Lie semi-simples complexes. C. R. Acad. Sci. Paris **238**, 437–439 (1954).

The author now applies his previous results on induced representations [see the preceding review] to the case where

G is a complex semi-simple Lie group none of whose simple factor groups is one of the exceptional simple Lie groups. Let Γ be the product of a Cartan subgroup H of G and the subgroup N corresponding to the negative roots with respect to H. Let $\hat{H}$ be the normalizer of H in G. The author states that he has verified for each of the complex classical Lie groups that each double coset of Γ in G meets $\hat{H}$; hence there are only a finite number of double cosets of Γ in G and they are in one-one correspondence with the Weyl group $\hat{H}/H$. Using this result the author studies the induced representations U^x of G where x is a one-dimensional unitary representation of $\Gamma=HN$ whose kernel contains N. The representations U^x are exactly the principal unitary representations of Gelfand and Naimark [see, e.g., Trudy Mat. Inst. Steklov. **36** (1950), pp. 1–150; these Rev. **13**, 722]. For the $n\times n$ complex unimodular group Gelfand and Naimark proved [see, e.g., loc. cit.] by induction on n that every U^x is irreducible. The present author outlines (without giving details) a proof that U^x is irreducible provided that the restriction of x to H is not invariant by any element $\neq 1$ of the Weyl group. Harish-Chandra has proved by an entirely different method without assuming that G is non-exceptional, that U^x is irreducible for all x outside some set of measure zero. The best possible result would be that U^x is irreducible for every x and that G can be an arbitrary (complex) semi-simple Lie group. This is apparently still unproved. *F. I. Mautner* (Baltimore, Md.).

Bruhat, François. Représentations induites des groupes de Lie semi-simples réels. C. R. Acad. Sci. Paris **238**, 550–553 (1954).

The author generalizes the methods and results of the preceding review to semi-simple Lie groups G which need not be complex. He assumes again that the simple factor groups of G are non-exceptional. He has verified now for all the (non-compact) classical groups that there is only a finite number of double cosets in G of the maximal solvable subgroup Γ. Using this, the author states the result that the principal unitary representation U^x is irreducible provided again that x is not invariant by any element of the Weyl group, and that $U^{x'}$ is equivalent to U^x if and only if x' and x are permuted by the Weyl group. No details of proofs are given. The problem arises again whether the above restrictions on x and G are necessary. *F. I. Mautner.*

van Est, W. T. Group cohomology and Lie algebra cohomology in Lie groups. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **56**=Indagationes Math. **15**, 484–492, 493–504 (1953).

Let $\mathfrak{M}$ be a Lie group, M its Lie algebra, V a finite-dimensional vector space over the field of real numbers, and π a linear representation of $\mathfrak{M}$ in V. In the literature of mathematics, various cohomology groups are introduced and studied, namely: (1) the n-dimensional cohomology group of $\mathfrak{M}$ as a topological space defined by means of exterior differential forms or otherwise, (2) the n-dimensional cohomology group of $\mathfrak{M}$ as a topological group defined by using the infinitely differentiable cochains analogous to the cochains of an abstract group, and (3) the n-dimensional cohomology group of the Lie algebra M. The common coefficient group is the vector space V on which the Lie group $\mathfrak{M}$ (and hence its Lie algebra M) operates by means of the linear representation π. Following the notations of the author, they are respectively denoted by $H(D_n)$, $H(C_n)$, and $H(K_n)$. All of them are vector spaces over the field of real numbers.

The main theorem established in these papers is that if $H(D_n)$ is trivial in the dimensions $n=0, 1, \cdots, k$, then $H(C_n)$ and $H(K_n)$ are canonically isomorphic for each $n=0, 1, \cdots, k$ whereas $H(C_{k+1})$ is isomorphic with a linear subspace of $H(K_{k+1})$. The proof of this theorem is built mainly on the study of a "bicochain complex" in which the cochains of $\mathfrak{M}$ and those of M are welded together.

Reviewer's remark: If the Lie group $\mathfrak{M}$ is compact, then this theorem is an immediate consequence of the known facts that, for every n, $H(C_n)$ is trivial and $H(K_n)$ is isomorphic canonically with a linear subspace of $H(D_n)$.

S. T. Hu (New Orleans, La.).

Borel, Armand. Homology and cohomology of compact connected Lie groups. Proc. Nat. Acad. Sci. U. S. A. **39**, 1142–1146 (1953).

The author determines the cohomology mod p (a prime), and to some extent that over the integers, of (a) the quotient groups of the classical groups by central subgroups, (b) the groups Spin (n), F_4, G_4. In addition some general results are given. Among the latter are the following: (1) If the cohomology ring $H^*(G, K_p)$ has a simple system of primitive generators, then the Pontryagin ring $H_*(G, K_p)$ is an anticommutative exterior algebra. This is so for all classical groups and all p. (2) A converse of the author's main theorem on transgression in universal bundles [Ann. of Math. (2) **57**, 115–207 (1953); these Rev. **14**, 490]: If the cohomology of the classifying space is a polynomial algebra, then that of the group is a Grassmann algebra; the two are connected by transgression.

We shall not attempt to list all the specific results. Generally speaking, these cohomology rings are not Grassmann algebras. For $n\geqq 7$, Spin (n) has torsion, all torsion coefficients are equal to 2. Its cohomology mod 2 is not generated by universally transgressive elements, the Pontryagin ring is not anticommutative, the cohomology of the classifying space is not a polynomial algebra. G_2 has 2-torsion, F_4 has 2- and 3-torsion; the cohomology over the integers is given. Steenrod squares and p-powers are indicated for Spin (n), G_2, F_4. The proofs (not given) make use of various fiberings of the groups involved, their spectral sequences, the cohomology of the cyclic groups etc. (in addition to known results, to be found in the author's paper mentioned above). Some of the results had been announced earlier [C. R. Acad. Sci. Paris **232**, 1628–1630 (1951); these Rev. **12**, 848]. *H. Samelson* (Princeton, N. J.).

Samelson, H. A class of complex-analytic manifolds. Portugaliae Math. **12**, 129–132 (1953).

The author proves the following result which is a generalization of a theorem of A. Borel [Ann. of Math. (2) **57**, 115–207 (1953), p. 197; these Rev. **14**, 490]: Let G be a compact Lie group and T a torus in G such that the dimension of the coset space G/T is even. Then G/T admits a complex structure which is invariant under operations of G. In the most essential case where G is semi-simple and $T=1$, the proof runs as follows: Let L be the Lie algebra of G and $\bar{L}$ the complex form of L. Using a Weyl basis for $\bar{L}$, the author chooses a suitable complex subalgebra M of $\bar{L}$ such that $\bar{L}$ is, as a real vector space, the direct sum of L and M and that the subgroups G' and S of the adjoint group $\bar{G}$ of $\bar{L}$ corresponding to L and M, respectively, have the properties $\bar{G}=G'S$, $G'\cap S=1$. Since $\bar{G}$ is a complex Lie group and S is a complex subgroup of $\bar{G}$, $\bar{G}/S$ has a natural complex structure invariant under operations of $\bar{G}$. This can be transferred on G' by the canonical homeomorphism $\bar{G}/S\to G'$

and gives a similar structure on G'. As G is locally isomorphic with G', G then also has a complex structure invariant under operations of G. *K. Iwasawa* (Cambridge, Mass.).

Badrikian, A. Structure de certains groupes localement compacts. Cahiers Rhodaniens **5**, 27–51 (1953).

The structure of the following types of topological groups is studied and some new proofs are given to known results on these groups: locally compact abelian groups; maximally almost periodic groups; locally compact groups with a fundamental system of neighborhoods of e, invariant under inner automorphisms, and groups with an invariant compact neighborhood of e; and the groups of outer automorphisms of compact groups. *K. Iwasawa* (Cambridge, Mass.).

Poncet, Jean. Une classe d'espaces homogènes possédant une mesure invariante. C. R. Acad. Sci. Paris **238**, 553–554 (1954).

Let H be a closed subgroup of a locally compact group G. Assume that for each neighborhood V of the identity there exists a neighborhood U of the identity such that $HU \subset VH$. The author shows that under these conditions the homogeneous space G/H of left cosets admits a measure which is invariant under G. *P. A. Smith* (New York, N. Y.).

NUMBER THEORY

***Kaprekar, D. R. Cycles of recurring decimals. Vol. II. (From $N=167$ to 213 and many other numbers.)** Privately printed, Devlali, 1953. 47 pp. 6 rupees.

Volume 1 of this work appeared in 1951 [see these Rev. **14**, 247]. The present volume extends the table of periodic decimals for M/N from $N=167$ to $N=213$. In connection with a discussion of these results there are given tables of the first 33 powers of 2 and 5 and a table of residue indices of 10 modulo p for p prime not exceeding 13709. The latter table is taken from M. Kraitchik, Recherches sur la théorie des nombres, [v. 1, Gauthier-Villars, Paris, 1924, pp. 131–139]. There is also a table of factors of $(10^e-1)/p$ where e is the exponent of 10 (mod p) for all primes $\leqq 61$ and for a half dozen larger primes. *D. H. Lehmer.*

Garcia, Mariano. On numbers with integral harmonic mean. Amer. Math. Monthly **61**, 89–96 (1954).

In a recent paper [same Monthly **55**, 615–619 (1948); these Rev. **10**, 284] Ore considered the harmonic mean $H(n)$ of the divisors of n and investigated the question of finding n for which $H(n)$ is an integer. He gave a list of the eleven numbers n less than 10,000 having this property. The author extends this investigation to the integers $<10^7$ and gives a complete list of 45 such numbers. Some 200 other very large numbers n were found including one having 35 prime factors and 85 49802 41758 61760 divisors with an integral harmonic mean of 11 36127 56336 64000. All perfect numbers have this property. It is proved that all numbers of the form p^aq, where p and q are distinct primes, which have the property are perfect. The methods of search for n's with integral $H(n)$ is not unlike those for multiplying perfect numbers. *D. H. Lehmer* (Berkeley, Calif.).

Piza, P. A. Fermat coefficients. Math. Mag. **27**, 141–146 (1954).

The author defines the coefficients (n, k) by

$$k(n, k) = \binom{2n-k}{k-1}$$

and studies their properties. Among other divisibility properties it is proved that $2n+1$ is a prime if and only if (n, k) is an integer for all k. The author fails to note the relation

$$(2n+1)(n, k) = (2n-1)\{(n-1, k)+2(n-1, k-1)\} - (2n-3)(n-2, k-2)$$

from which it follows by induction that $(2n+1)(n, k)$ is an integer for all n and k. This fact is useful in proving the theorems announced by the author. There is a table of (n, k) for all integers $k \leqq n \leqq 17$. *D. H. Lehmer.*

Rodeja, F., E. G.-. On the diophantine equation $x^3+y^3=2z^3$. Revista Mat. Hisp.-Amer. (4) **13**, 229–240 (1953). (Spanish)

The equation $x^3+y^3=2z^3$ is solved completely in the cases $(x, y)=1$ or 2, three sets of formulas being given in the first case and four in the second. Each set of formulas is in terms of two relatively prime parameters. An earlier set of formulas for the case $(x, y)=1$ by Alef [same Revista (4) **12**, 3–8 (1952); these Rev. **14**, 20] was noted to be incomplete by this reviewer. *I. Niven* (Eugene, Oreg.).

Kanold, Hans-Joachim. Über befreundete Zahlen. I. Math. Nachr. **9**, 243–248 (1953).

The author conjectures that there does not exist a pair of amicable numbers m_1 and m_2 where one of the numbers, say m_1, is a power of a prime p. The following results are proved. If m_1 and m_2 are amicable and $m_1=p^a$, then $m_1>m_2>10^{1400}$ and $a>1400$. Moreover, the number of different prime factors of m_2 is greater than 300. (Some steps of the proof can be slightly simplified. For instance, the equation (6) is impossible (mod 4) if $a>1$.) *A. Brauer.*

Kanold, Hans-Joachim. Über befreundete Zahlen. II. Math. Nachr. **10**, 99–111 (1953).

This paper is a continuation of the preceding paper. In it some necessary conditions for amicable numbers m_1 and m_2 are proved. If both m_1 and m_2 are divisible only by two different primes, then m_1 and m_2 are equal, even, perfect numbers. If m_1 is even, m_2 odd, and $(m_1, m_2)=1$, then $m_1m_2 \equiv 2 \pmod{24}$. If, moreover, $m_2<m_1$, then m_2 is divisible by at least 5 different primes. *A. Brauer.*

Rohrbach, Hans, und Volkmann, Bodo. Zur Theorie der asymptotischen Dichte. J. Reine Angew. Math. **192**, 102–112 (1953).

This paper contains a discussion and exposition of a theorem of Ostmann on the asymptotic density of the sum of two sets of integers [J. Reine Angew. Math. **187**, 183–188 (1950); these Rev. **11**, 646]. The authors succeed in elucidating and in considerably simplifying Ostmann's proof. The paper is independent of a paper by Martin Kneser [Math. Z. **58**, 459–484 (1953); these Rev. **15**, 104], who derived Ostmann's result using methods related to those introduced by the reviewer. *H. B. Mann.*

Sandham, H. F. Two series of partitions. Amer. Math. Monthly **61**, 104–106 (1954).

The series are

$$\sum_0^\infty \frac{p(n)}{(n-1/24+\theta)^{\frac{1}{4}}} \frac{\sinh 2\pi(\frac{2}{3}(n-1/24+\theta))^{\frac{1}{2}}}{\cosh 3\pi(\frac{2}{3}(n-1/24+\theta))^{\frac{1}{2}}} = \frac{1}{2^{\frac{1}{2}}\pi\theta}$$

and

$$\sum_0^\infty \frac{p(n)(n-\theta)}{\cosh \pi(2n-\frac{1}{4}+4\theta)^{\frac{1}{2}}}=0 \quad (\theta>0),$$

where $p(n)$ is the number of partitions. The formulas are proved by termwise integration of identities of Euler and Jacobi. *L. Carlitz* (Durham, N. C.).

Carlitz, L. A theorem of Ljunggren and Jacobsthal on Bernoulli numbers. Proc. Amer. Math. Soc. **5**, 34–37 (1954).

The main result of this note is the formula

$$\sum_{r=1,\,(r,m)=1}^{m}(-1)^r E_k\left(\frac{r}{m}\right)=m^{1-k}E_k(0)\prod_{p\mid m}(1-p^{k-1}),$$

where m is odd, $E_k(x)$ is the Euler polynomial, and the product extends over all primes p dividing m. There is a similar formula in which r is restricted to the range $1\leq r\leq\frac{1}{2}(m-1)$, m is odd and ≥ 3. The proofs are modeled on proofs by Ljunggren [Avh. Norske Vid. Akad. Oslo. I. 1947, no. 5 (1948); these Rev. **9**, 568] and Jacobsthal [Norske Vid. Selsk. Forh., Trondheim **22**, no. 24, 107–112 (1950); these Rev. **11**, 581] of corresponding theorems involving Bernoulli numbers. The note also contains generalizations of these results which hold for Bernoulli and Euler polynomials of higher order. *A. L. Whiteman.*

Carlitz, L. Some hypergeometric congruences. Portugaliae Math. **12**, 119–128 (1953).

Employing certain results on hypergeometric series [see Bailey, Generalized hypergeometric series, Cambridge, 1935, pp. 25–28, 86] the author derives a series of formulas of which the following is typical:

$$\sum_{r=0}^{m}(4r+1)\left(\frac{1\cdot 3\cdots(2r-1)}{2\cdot 4\cdots 2r}\right)^3\equiv -2\binom{2m}{m}^3 \pmod{4m+3},$$

where the modulus is a prime. These formulas may be thought of as congruential analogues of various formulas of Ramanujan [see Hardy, Ramanujan, Cambridge, 1940, Ch. 7; these Rev. **3**, 71]. *A. L. Whiteman.*

Rankin, R. A. A minimum problem for the Epstein zeta-function. Proc. Glasgow Math. Assoc. **1**, 149–158 (1953).

Let $Z_k(s)=\sum_{m,n=-\infty}^{\infty}{}' h(m,n)^{-s}$, where

$$h(m,n)=am^2+2dmn+bn^2,$$

a positive definite quadratic form of determinant unity, and $Z_Q(s)=6(\frac{1}{2}\sqrt{3})^s\zeta(s)L(s)$. The author proves that for all $s\geq 1.035$, the inequality $Z_k(s)\geq Z_Q(s)$ holds, with equality only when k and Q are equivalent forms. The Epstein zeta-function $Z_k(s)$ is a constant multiple of the mean value of the variance of the number of lattice points in a random circle [see D. G. Kendall and R. A. Rankin, Quart. J. Math., Oxford Ser. (2) **4**, 178–189 (1953); these Rev. **15**, 237]. The result above asserts that the minimum variance is associated with a hexagonal lattice. *R. Bellman.*

Fogel', È. K. An elementary proof of formulae of de la Vallée Poussin. Latvijas PSR Zinātņu Akad. Vestis **1950**, no. 11 (40), 123–130 (1950). (Russian. Latvian summary)

The author discusses by elementary methods, involving only finite sums and the properties of a^x, $\log x$, $\sin x$, $\cos x$, the asymptotic relations

$$\sum_{\substack{p\leq x\\ p\equiv l(k)}}1\sim\frac{x}{h\log x},\qquad \sum_{\substack{n\leq x\\ n\equiv l(k)}}\lambda(n)=o(x),\qquad \sum_{\substack{n\leq x\\ n\equiv l(k)}}\mu(n)=o(x),$$

where $(l,k)=1$, $h=\varphi(k)$, p runs through primes, and $\lambda(n)$ and $\mu(n)$ are the functions of Liouville and Möbius. The proof of the first formula follows the lines of A. Selberg's elementary proof of the prime number theorem (the special case $k=1$). A substantial part of the argument is taken up with a proof, according to the author's canons, of the formula

$$\sum_{\substack{p\leq x\\ p\equiv l(k)}}\frac{\log p}{p}=\frac{1}{h}\log x+O(1)$$

of Mertens; this involves the use of real and complex characters. (In a footnote the author acknowledges Selberg's treatment of the general case $k\geq 1$ by the use of real characters, his own work having been done in ignorance of this.) It is stated that the second and third results can be deduced from the first by methods used in a previous paper by the author [Latvijas PSR Zinātņu Akad. Fiz. Mat. Inst. Raksti. **2**, 14–45 (1950); these Rev. **13**, 824]. The concluding section is devoted to analogues concerning primes representable by a positive definite binary quadratic form.

A. E. Ingham (Cambridge, England).

Pyateckiĭ-Šapiro, I. I. On the distribution of prime numbers in sequences of the form $[f(n)]$. Mat. Sbornik N.S. **33(75)**, 559–566 (1953). (Russian)

The principal result is that the number of primes in the sequence $[n^c]$ not exceeding x is asymptotically $x^{1/c}/\log x$ for $1\leq c<12/11$. The proof depends upon the estimation of trigonometric sums à la Vinogradoff. *R. Bellman.*

Koecher, Max. Ein neuer Beweis der Kroneckerschen Grenzformel. Arch. Math. **4**, 316–321 (1953).

Let $\tau=x+iy$ be a complex number, where $y>0$ and let

$$\zeta(\tau;s)=\sum_{m,n}{}'|m\tau+n|^{-2s}\quad(\mathrm{Re}(s)>1),$$

where (m,n) runs through all pairs of integers $\neq(0,0)$. Then if $\eta(\tau;s)=\pi^{-1}y\zeta(\tau;s)-(s-1)^{-1}$, the formula in question is

$$\eta(\tau;1)=-\tfrac{1}{12}\log\left(\Delta(\tau)\Delta(-\bar{\tau})\right)-2\log y+K,$$

where K is a constant ($=2C-2\log 2$, where C is Euler's constant) and $\Delta(\tau)=e^{2\pi i\tau}\prod_{n=1}^{\infty}(1-e^{2\pi i n\tau})^{24}$. The author's proof is based on the following well known theorem: If $f(\tau)$ is regular and $\neq 0$ for $y>0$, $f(\tau)=f(-\bar{\tau})$, $y^m|\log|f(\tau)||$ is bounded for some $m>0$, and if $(c\tau+d)^{-12}f(a\tau+b/c\tau+d)=f(\tau)$ for all modular substitutions $\tau\to a\tau+b/c\tau+d$, then

$$f(\tau)=c\Delta(\tau),$$

where c is a constant. The constant K is determined by taking $\tau=i$ and applying Hurwitz's formula

$$\Delta(i)=(2\pi)^{-24}\Gamma^{24}(\tfrac{1}{4}).$$

H. D. Kloosterman (Leiden).

Zulauf, Achim. Über den dritten Hardy-Littlewoodschen Satz zur Goldbachschen Vermutung. J. Reine Angew. Math. **192**, 117–128 (1953).

The author uses Linnik's modification [Mat. Sbornik **19(61)**, 3–8 (1946); these Rev. **8**, 317] of the Hardy-Littlewood method to obtain a new proof of a generalization of the theorem that almost all even positive integers are sums of two primes. *T. Estermann* (London).

Zulauf, Achim. Beweis einer Erweiterung des Satzes von Goldbach-Vinogradov. J. Reine Angew. Math. **190**, 169–198 (1952).

Let $m\geq 3$ and $K\geq 1$ be given and let $a_1,\cdots,a_m$ be relatively prime to K. With fixed m, K, $a_1,\cdots,a_m$ let $N(n)$ be

the number of solutions in odd primes $p_\tau \equiv a_\tau \pmod K$ of the equation $n=p_1+\cdots+p_m$. Then there is a singular series $S(n)$ such that for all large $n \equiv m \pmod 2$ such that $n \equiv a_1+\cdots+a_m \pmod K$ we have

$$N(n)=\frac{n^{m-1}}{\log^m n}\,\frac{S(n)}{(m-1)!\,\phi^m(K)}\{1+o(1)\}$$

and $0<c<S(n)/K<c'$ for suitable absolute constants c and c'. Actually, by picking $\delta=(m+1)(\log\log n)/\log n$ in the author's proof, the error term $o(1)$ can be replaced by $O(\log\log n/\log n)$. This proof replaces an earlier one by H. Rademacher [Math. Z. **25**, 627–657 (1926)] who, however, assumed a modified Riemann hypothesis. The case $m=3$ has recently been dealt with by a number of authors [K. Iseki, Proc. Japan Acad. **25**, 185–187 (1949); these Rev. **15**, 102; R. Ayoub, Trans. Amer. Math. Soc. **74**, 482–491 (1953); these Rev. **14**, 847; I. Földes, C. R. 1ier Congrès Math. Hongrois, 1950, Akad. Kiadó, Budapest, 1952, pp. 473–492; these Rev. **15**, 13]. Unlike these authors who use Vinogradov's results on exponential sums with prime summation letter, Zulauf uses a method of Linnik and Čudakov. The author closely follows Čudakov [Ann. of Math. (2) **48**, 515–542 (1947); these Rev. **9**, 11]. *L. Schoenfeld.*

Zulauf, Achim. Zur additiven Zerfällung natürlicher Zahlen in Primzahlen und Quadrate. Arch. Math. **3**, 327–333 (1952).

Let $K_\sigma, a_\sigma, b_\tau$ be fixed positive integers with $(K_\sigma, a_\sigma)=1$ for $\sigma=1,\cdots,s$ and $\tau=1,\cdots,t$. Let $s\geqq 1$, $t\geqq 0$ and $\omega=s+\frac{1}{2}t\geqq 2$. Let $N_{s,t}(n)$ be the number of solutions in odd primes $p_\sigma \equiv a_\sigma \pmod{K_\sigma}$ and integers g_τ of the equation

$$n=p_1+\cdots+p_s+b_1g_1^2+\cdots+b_tg_t^2.$$

The author briefly sketches the proof of three results. The first is that if $\omega>2$ and n is large then

$$N_{s,t}(n)=\frac{n^{\omega-1}}{\log^s n}\mathfrak{S}_{s,t}(n)+O\left(\frac{n^{\omega-1}\log\log n}{\log^{s+1} n}\right).$$

This result, proved along the lines of the preceding paper, includes the result of the above paper which corresponds to the case $t=0$ and $s=m$; it also includes a result of H. Halberstam [Proc. London Math. Soc. (2) **53**, 363–380 (1951); these Rev. **13**, 112] in which all $K_\sigma, a_\sigma, b_\tau$ are 1. The second result deals with the case $\omega=2$ for "almost all" n. The third result gives lower bounds for the singular series $\mathfrak{S}_{s,t}(n)$ depending on whether $\omega>2$ or $\omega=2$.

L. Schoenfeld (Urbana, Ill.).

Krull, Wolfgang. Über Polynomzerlegung mit endlich vielen Schritten. II. Math. Z. **59**, 296–298 (1953).

In a preceding note [Math. Z. **59**, 57–60 (1953); these Rev. **15**, 96] the author inquired about the existence of an infinite sequence $K_1, K_2, \cdots$ of distinct algebraic number fields of finite degree over the field K_0 of rational numbers with the following property (Z): For each irreducible polynomial $p \in K_0[x]$ there is a number $v^*(p)>0$, which can be computed in a finite number of rational operations, such that if p is irreducible over K_i $(1\leqq i\leqq v^*(p))$ then p is irreducible over every K_i. He showed in that preceding note that the sequence of imaginary quadratic number fields has property (Z). In the present note he uses some elementary facts to establish two sufficient conditions for a sequence $K_1, K_2, \cdots$ to have property (Z), and then uses these to find many examples. *E. R. Kolchin* (New York, N. Y.).

Cohen, Eckford. Congruence representations in algebraic number fields. Trans. Amer. Math. Soc. **75**, 444–470 (1953).

In two earlier papers [Duke Math. J. **19**, 115–129 (1952); C. R. Acad. Sci. Paris **234**, 787–788 (1952); these Rev. **13**, 823] the author has considered general arithmetic functions, defined on residue class rings in rational and algebraic fields. In this paper he gives some applications of the general theory to special arithmetic functions such as Rademacher sums, Ramanujan sums, Hecke sums and some sums that are related to Hecke sums. Then he uses these sums in order to get the number of solutions x_{ij} of congruences of the form $\rho \equiv \alpha_1 x_{11}\cdots x_{1j_1}+\cdots+\alpha_s x_{s1}\cdots x_{sj_s} \pmod A$, where ρ is an arbitrary algebraic integer, α_i $(i=1,\cdots,s)$ are given algebraic integers and A is an ideal in an algebraic field. In particular, the number of the solutions of the quadratic congruences $\rho \equiv \alpha_1\xi_1^2+\cdots+\alpha_s\xi_s^2 \pmod A$ are given, when A is an odd prime and $\alpha_1,\cdots,\alpha_s$ are prime to A. All these applications depend on a theorem, which states that every arithmetic function considered here is a linear function of Hecke's exponential functions. *H. Bergström.*

Carlitz, L. Note on the cyclotomic polynomial. Amer. Math. Monthly **61**, 106–108 (1954).

Let p denote a prime $\geqq 3$ and let X denote the cyclotomic polynomial $(x^p-1)/(x-1)$. It is well known that $4X=Y^2-(-1)^{(p-1)/2}pZ^2$, where Y, Z are polynomials in x with integral coefficients. The author puts $x=t+1$, $U(t)=Y(t+1)$, $V(t)=Z(t+1)$ and derives an algorithm for computing the coefficients of $U(t)$ and $V(t)$ (mod p). He also obtains an explicit formula for the residue of the coefficients of $V(t)$ in terms of Bernoulli numbers of higher order.

A. L. Whiteman (Princeton, N. J.).

Carlitz, Leonard. Weighted quadratic partitions over a finite field. Canadian J. Math. **5**, 317–323 (1953).

Assume $\alpha \in GF(q)$, $q=p^n$, p an odd prime, and write $e(\alpha)=\exp(2\pi i t(\alpha)/p)$, where $t(\alpha)=\alpha+\alpha^p+\cdots+\alpha^{p^{n-1}}$. The author applies known results on Gauss sums in a finite field [Carlitz, Duke Math. J. **14**, 1105–1120 (1947); these Rev. **9**, 337] to evaluate the sum

$$S=\sum_{a_1\xi_1^2+\cdots+a_s\xi_s^2=\alpha} e(2\lambda_1\xi_1+\cdots+2\lambda_s\xi_s);$$

a_i, λ_i fixed, sum for ξ's,

explicitly or in terms of Kloosterman sums:

$$K(\alpha,\omega)=\sum_{\beta\neq 0} e(\alpha\beta+\omega/\beta), \qquad \alpha\neq 0,$$

$$L(\alpha,\omega)=\sum_{\beta\neq 0}\psi(\beta)e(\alpha\beta+\omega/\beta), \quad \alpha\neq 0,$$

where $\psi(\beta)=1$ or -1 according as β is a square or non-square in $GF(q)$. For example, if $s=2t$, $\alpha\neq 0$, $\omega\neq 0$,

$$S=q^{s-1}l+q^{t-1}\psi((-1)^t\delta)K(\alpha,\omega),$$

where $\delta=a_1\cdots a_s$, $l=1$ if all $\lambda_i=0$, $l=0$ otherwise. He obtains bounds for S, for example, if $s=2t$,

$$|S-q^{s-1}l|<2q^{t-1}.$$

He also studies sums similarly related to non-singular quadratic forms $Q=\sum a_{ij}u_iu_j$;

$$S(\alpha,\lambda,Q)=\sum_{Q(\xi)=\alpha} e(2\lambda_1\xi_1+\cdots+2\lambda_s\xi_s),$$

for which the reciprocity formula is proved:

$$S(\alpha,\lambda,Q)=S(\alpha,\lambda',Q'),$$

where Q' is the inverse form to Q, and the λ_i' are determined

by $\lambda_i=\sum\alpha_{ij}\lambda_j'$. For the forms Q, which can always be congruently reduced to diagonal form over $GF(q)$, the number l has the same significance as above. *R. Hull.*

Carlitz, L. A reciprocity formula for weighted quadratic partitions. Math. Scand. 1, 286–288 (1953).

Let $q=p^n$, where p is an odd prime. For $\alpha \in GF(q)$ let $t(\alpha)$ denote the trace of α and put $e(\alpha)=\exp(2\pi i t(\alpha)/p)$. Also let

$$(*) \qquad S(\alpha,\lambda,Q)=\sum_{Q(\xi)=\alpha} e(2\lambda_1\xi_1+\cdots+2\lambda_r\xi_r),$$

where $Q(u)=\sum_{i,j}'\alpha_{ij}u_iu_j$ ($\alpha_{ij}, \lambda_j \in GF(q)$, $\delta=|\alpha_{ij}|\neq 0$), and the summation in the right member of (*) is over all $\xi_i \in GF(q)$ such that $Q(\xi_1,\cdots,\xi_r)=\alpha$. In the paper reviewed above the author showed incidentally that if $\lambda_k=\sum_{j=1}^{r}{}'\alpha_{kj}\lambda_j'$ ($k=1,\cdots,r$), then the reciprocity relation

$$S(\alpha,\lambda,Q)=S(\alpha,\lambda',Q')$$

is satisfied, where $Q'(u)$ denotes the quadratic form inverse to $Q(u)$. In this note he gives a direct proof of this result as well as some extensions. He also considers the analogous formula when the coefficients are rational integers.

A. L. Whiteman (Princeton, N. J.).

Ayoub, R. G. On the Waring-Siegel theorem. Canadian J. Math. 5, 439–450 (1953).

Let F be a totally real algebraic number field of degree n and with discriminant D. Let further

$$\varphi(\xi)=\alpha\xi^k+\alpha_1\xi^{k-1}+\cdots+\alpha_k$$

be a polynomial of degree k with integral coefficients α, $\alpha_i>0$ ($\alpha>\beta$ means that $\alpha^{(j)}>\beta^{(j)}$ for all conjugates $\alpha^{(j)}$, $\beta^{(j)}$ of α, β respectively). By means of Siegel's generalization of the method of Hardy and Littlewood the author proves the following asymptotic formula for the number $A(\nu)$ of solutions of $\nu=\varphi(\xi_1)+\cdots+\varphi(\xi_s)$ (where $\nu>0$) in integers ξ_i of F:

$$A(\nu)=D^{k(1-s)}\sigma_{n,k,s}\left[\frac{\Gamma^s\left(1+\frac{1}{k}\right)}{\Gamma\left(\frac{s}{k}\right)}\right]^n N(\alpha)^{-s/k}N(\nu)^{-1+s/k}+o(N(\nu)^{-1+s/k}),$$

where $s\geqq n(2^k+n)+1$ and $\sigma_{n,k,s}$ is the "singular series" (N denotes the norm in F). The discussion of the singular series is deferred to a later paper. *H. D. Kloosterman.*

Iseki, Kanesiroo. On a general divisor problem in algebraic number fields. Nat. Sci. Rep. Ochanomizu Univ. 4, 1–21 (1953).

Let $v(n)$ denote the number of integral solutions of $u^2+v^2=n$. Hardy gave the identity

$$(1) \qquad \sum_{n\leq x}{}'U'(n)=\pi x-1+x^{\frac{1}{2}}\sum_{n=1}^{\infty}\frac{U(n)}{n}J_1(2\pi(nx)^{\frac{1}{2}})$$

(the prime attached to $\sum$ means that, if x is an integer, the last term $v(x)$ is to be halved, J_1 denotes the Bessel function of order 1). (Later Hardy and Landau gave another identity of a similar form but with the sum in the right side of (1) only extended to a finite number of terms and the remainder involving Bessel functions and an infinite sum, where the term of order n has the factor $v(n)/n$.) The author now gives a generalization of these identities, considering the number of ideals of given norms in a more general algebraic number field. Let k_j be for each j, $j=1,2,\cdots,r$, an algebraic number field of degree n_j and put $N=n_1+n_2+\cdots+n_r$ ($N>2$). Denoting by $\xi_j(s)$ the Dedekind ζ-function and putting

$$Z(s)=\zeta_1(s)\cdots\zeta_r(s)=\sum_{n=1}^{\infty}F(n)n^{-s},$$

he can give Hardy's identity the following general form:

$$\sum_{n\leq x}F(n)(x-n)^M=\operatorname{Res}_{s=1}\left(\frac{Z(s)M!x^{s+M}}{s\cdots(s+M)}\right)+Z(0)x^M+\sum_{n=1}^{\infty}\frac{F(n)}{n^{M+1}}\frac{M!}{2\pi i}\int_{(1)}\frac{Z(s)}{Z(1-s)}\frac{(nx)^{s+M}}{s\cdots(s+M)}ds,$$

where $M=(N-1)/2$. In the same way Hardy-Landau's identity is generalized. *H. Bergström* (Göteborg).

Iwasawa, Kenkichi. On solvable extensions of algebraic number fields. Ann. of Math. (2) 58, 548–572 (1953).

Let k_0 be a finite algebraic number field, k/k_0 its maximal abelian extension and Σ/k_0 its maximal solvable extension. The author proves that the Galois group of Σ/k is a fixed group, completely independent of the choice of k_0.

Let S be any class of finite groups, called S-groups, which satisfies the following conditions: subgroups and products of S-groups are S-groups; homomorphic images of S-groups are S-groups. A compact group is called an S-group if it is a projective limit of finite S-groups. There exists a "universal" S-group $F(S)$ of which every compact separable S-group is a homomorphic image. The group $F(S)$ may be described as follows: let F be a free group on countably many generators; let H be the family of all normal subgroups M of F whose factor groups are finite S-groups and which contain almost all of the generators of F; then $F(S)$ is the projective limit of the family of the factor groups F/M, M in H, with the homomorphisms defined in the obvious manner. The group $F(S)$ may also be characterized, among all separable compact S-groups, by the following property: let G be any finite S-group, N a normal subgroup of G and φ an isomorphism of G/N with $F(S)/V$, where V is a normal open subgroup of $F(S)$; then there exist an open normal subgroup W of $F(S)$ contained in V and an isomorphism ψ of G with $F(S)/W$ such that $\psi(N)=V/W$ and that φ is the isomorphism deduced from ψ by going over to the quotients. A Galois extension of a field k is called an S-extension if its Galois group is an S-group. There is a largest S-extension M/k of k; the author first determines the condition under which the Galois group of the largest S-extension of k is the universal S-group $F(S)$. In order to formulate this condition, he introduces the following definition: let k be a field, K/k a Galois extension of k, G a finite group, N a normal subgroup of G and φ an isomorphism of G/N with the Galois group of K/k; then the author denotes by $P(K/k, G/N, \varphi)$ the problem of finding a Galois extension L/k containing K/k and an isomorphism ψ of G with the Galois group of L/k such that $\psi(N)$ is the Galois group of L/K and φ the isomorphism deduced from ψ by Galois theory. This being said, the group of the maximal S-extension of k is $F(S)$ if and only if it is separable and every imbedding problem $P(K/k, G/N, \varphi)$ where G is an S-group has a solution.

Now assume that k/k_0 is the maximal abelian extension of a finite algebraic number field k_0 and that S is the class of solvable finite groups. Then the Galois group of the maximal solvable extension of k is proved to be $F(S)$. In order to do this, the author shows that every problem $P(K/k, G/N, \varphi)$ where G is solvable has a solution. He first

reduces the problem to the case where N is an elementary p-group, p a prime. One of the results he obtains in the course of his proof is the following. Let p be a prime and k a field of characteristic $\neq p$ containing a primitive pth root of unity. In order for $P(K/k, G/N, \varphi)$ to have a solution whenever G is a p-group, it is necessary and sufficient that the following conditions be satisfied: (1) if K/k is any Galois extension of k whose Galois group $G(K/k)$ is a p-group, the second cohomology group of $G(K/k)$ in the multiplicative group of K is trivial; (2) if k^* is the multiplicative group of k and k^{*p} the group of pth powers of k^*, then k^*/k^{*p} is infinite. This in turn is derived from a lemma which states a necessary and sufficient condition for a certain individual problem $P(K/k, G/N, \varphi)$ to have a solution (under certain assumptions on K/k and N; in particular, N is a p-group); this condition is too long to be given here. *C. Chevalley* (Nagoya).

Nakano, Noboru. Über den Fundamentalsatz der Idealtheorie in unendlichen algebraischen Zahlkörpern. J. Sci. Hiroshima Univ. Ser. A. **15**, 171–175 (1952).

The author considers a commutative integral domain $\mathfrak{O}$ with unity for which the following two conditions are satisfied: (1) The prime ideals have no proper divisors. (2) If $\mathfrak{a} \neq \mathfrak{O}$ is an ideal of $\mathfrak{O}$, if $\mathfrak{h}$ is its radical, and if $\mathfrak{p}$ is a prime ideal divisor of $\mathfrak{a}$, then for every $\pi \in \mathfrak{p}$, there exists an element s of $\mathfrak{O}$ such that $s\pi$ as $\mathfrak{h}$, s non-$\in \mathfrak{p}$. It is shown that then every ideal $\mathfrak{a}$ of $\mathfrak{O}$ can be represented as an intersection of primary ideals $\mathfrak{q}$. To every prime ideal divisor $\mathfrak{p}$ of $\mathfrak{a}$, there corresponds one such primary component $\mathfrak{q}$, the isolated primary component belonging to $\mathfrak{p}$, and $\mathfrak{q}$ is the intersection of those primary ideals of $\mathfrak{O}$ belonging to $\mathfrak{p}$ which divide $\mathfrak{a}$. If $\mathfrak{O}$ is the ring of algebraic integers in an (infinite) algebraic number field, the conditions (1) and (2) are satisfied. In this particular case, the result was proved by W. Krull [Math. Z. **29**, 42–54 (1928)]. *R. Brauer.*

Nakano, Noboru. Idealtheorie in einem speziellen unendlichen algebraischen Zahlkörper. J. Sci. Hiroshima Univ. Ser. A. **16**, 425–439 (1953).

In the case of infinite algebraic number fields K, it may happen that there exist idempotent prime ideals $\mathfrak{p}$, $\mathfrak{p} = \mathfrak{p}^2$. The author studies a class of fields in which this possibility does not arise. He considers a field K which is the union of an ascending sequence $\{K_\nu\}$ of fields of finite degrees over the field of rational numbers such that the following two conditions are satisfied for every rational prime p. (I) For sufficiently large ν, all the prime ideal factors $\mathfrak{p}_{\nu i}$ of p in K_ν remain unramified in $K_{\nu+1}$. (II) For every ν, there exists a $\mu > \nu$ such that every $\mathfrak{p}_{\nu i}$ factors non-trivially in K_μ. There exist fields of this type. For instance, one may take the field obtained by adjoining the pth roots of unity for all prime numbers p.

A number of properties of the fields K satisfying the conditions are obtained. For instance, if R is the ring of algebraic integers in K, no ideal $\mathfrak{a}$ $(\neq (0), R)$ of R is idempotent. An ideal $\mathfrak{q}$ is a primary ideal of R belonging to a prime ideal $\mathfrak{p}$, if and only if $\mathfrak{q}$ is a power of $\mathfrak{p}$. Two powers $\mathfrak{p}^a$ and $\mathfrak{p}^b$ of a prime ideal are equal only if $a = b$. Further, $\mathfrak{p}^{a+n} : \mathfrak{p}^a = \mathfrak{p}^n$ for positive exponents a and n. If an ideal $\mathfrak{a}$ $(\neq (0), R)$ is written as the intersection of its isolated primary components $\mathfrak{q}$ [cf. Krull, Math. Z. **29**, 42–54 (1928); and the paper reviewed above] and if $\mathfrak{q} = \mathfrak{p}^e$ when $\mathfrak{p}$ is the corresponding prime ideal, then for a fixed $\mathfrak{a}$ all the exponents e lie below a fixed bound.

A fixed component $\mathfrak{q}$ of $\mathfrak{a}$ is redundant in the representation of $\mathfrak{a}$ as intersection of its isolated primary components, if and only if $\mathfrak{a} : \mathfrak{p} = \mathfrak{a}$ for the prime ideal $\mathfrak{p}$ belonging to $\mathfrak{q}$. If $\mathfrak{a} : \mathfrak{p} \neq \mathfrak{a}$ for every prime ideal divisor $\mathfrak{p}$ of $\mathfrak{a}$, then, for every divisor $\mathfrak{b}$ of $\mathfrak{a}$, there exists a factorization $\mathfrak{a} = \mathfrak{b}\mathfrak{c}$ where $\mathfrak{c}$ is an ideal of R. If $\mathfrak{p}$ is a prime ideal divisor of $\mathfrak{a}$ and if $\mathfrak{a} : \mathfrak{p} = \mathfrak{a}$, this is no longer true for every divisor $\mathfrak{b}$ of $\mathfrak{a}$. For instance, it is not true for $\mathfrak{b} = \mathfrak{p}$. If we write $\mathfrak{a}^{-1}$ for $R : \mathfrak{a}$, then $\mathfrak{p}^{-1} = R$ for every prime ideal $\mathfrak{p}$ of R. More generally, $\mathfrak{a}^{-1} = R$, if, in the representation of the ideal $\mathfrak{a}$ as the intersection of its isolated primary components, each component $\mathfrak{q}$ is irredundant. *R. Brauer* (Cambridge, Mass.).

Nakano, Noboru. Über idempotente Ideale in unendlichen algebraischen Zahlkörpern. J. Sci. Hiroshima Univ. Ser. A. **17**, 11–20 (1953).

This paper is a continuation of the two papers reviewed above. If K is an (infinite) algebraic number field, it is shown that an ideal $\mathfrak{a}$ $(\neq (0), (1))$ is idempotent, if and only if its isolated primary components are idempotent prime ideals. A prime ideal $\mathfrak{p}$ is idempotent if and only if the intersection of $\mathfrak{p}$ with a subfield K_1 of K of finite degree over the field K_0 of rational numbers has arbitrarily high ramification in suitable fields K_2 of finite degree over K_0 with $K_1 \subset K_2 \subset K$. If $\mathfrak{p} \neq \mathfrak{p}^2$, the intersection of all the powers of $\mathfrak{p}$ is (0). Further, the primary ideals belonging to $\mathfrak{p}$ are the powers of $\mathfrak{p}$ and there do not exist proper intermediate ideals between two successive powers of $\mathfrak{p}$. If $\mathfrak{p} = \mathfrak{p}^2$, there exist primary ideals belonging to $\mathfrak{p}$ which are different from $\mathfrak{p}$. *R. Brauer.*

Nakano, Noboru. Über die kürzeste Darstellung der Ideale im unendlichen algebraischen Zahlkörper. J. Sci. Hiroshima Univ. Ser. A. **17**, 21–25 (1953).

Every ideal $\mathfrak{a}$ of an infinite algebraic number field $\mathfrak{R}$ can be written as the intersection of its isolated primary components $\mathfrak{q}$ [see, e.g., the paper reviewed third above]. It is shown that none of these components $\mathfrak{q}$ contains the intersection of the other ones, if and only if $\mathfrak{a}$ has only a finite number of prime ideal divisors. An equivalent condition is that, for every isolated primary component $\mathfrak{q}$ of $\mathfrak{a}$, there exists an integer r in $\mathfrak{R}$ such that $\mathfrak{a} : (r) = \mathfrak{q}$. *R. Brauer.*

Aigner, Alexander. Zur einfachen Bestimmung der Klassengruppe eines imaginär quadratischen Körpers. Arch. Math. **4**, 408–411 (1953).

A method is discussed for obtaining information about odd prime factors and powers of $2 (\geqq 8)$ in the invariants of the class group of an imaginary quadratic field. The fields generated by $\sqrt{-179}$, $\sqrt{-265}$, $\sqrt{-383}$, $\sqrt{-562}$, $\sqrt{-705}$ are treated in detail. They have class groups of types (5), (4, 2), (17), (8), (6, 2, 2). *O. Taussky-Todd.*

Kuipers, L., and Meulenbeld, B. On a certain classification of the convergents of a continued fraction. I. Nieuw Arch. Wiskunde (3) **1**, 199–211 (1953).

This paper is Part I of an investigation concerning the sequence of approximants of a simple continued fraction. The authors first note that any irreducible fraction P/Q falls into one of 8 classes determined by the ordered pair $(r(P), r(Q))$ of remainders mod 3 of P, Q. The possible classes of two and of three consecutive approximants are tabulated and it is noted that if the partial denominators of the simple continued fraction satisfy $r(a_{n+j}) = k$, $j = 2, 3, \cdots, 7$, for either $k = 1$ or $k = 2$, then the approximants P_{n+j}/Q_{n+j}, $j = 0, 1, \cdots, 7$, yield a sequence of 8 different classes. The labor involved in determining the

class of a given approximant P_n/Q_n is materially reduced by means of a reduction process. *W. T. Scott.*

Korobov, N. M. Multidimensional problems of the distribution of fractional parts. Izvestiya Akad. Nauk SSSR. Ser. Mat. **17**, 389–400 (1953). (Russian)

In this paper the author develops a new method for studying problems of uniform distribution of fractional parts of functions. Results previously obtained by him by several different methods are deduced and improved.

Thus Theorem 1 states that if $f(x)$ is an arbitrary polynomial with integral coefficients, not identically zero, and if $\lambda>1$ is an algebraic integer satisfying a certain condition, then the function $\alpha\lambda^x f(x)$ is uniformly distributed, where

$$\alpha=\sum_{i=1}^{\infty}\frac{\varphi^{(i)}\gamma_i}{p_i(\lambda^{\tau_i}-1)}\left(\frac{1}{\lambda^{n_i}}-\frac{1}{\lambda^{n_{i+1}}}\right).$$

Here the numbers p_i, n_i, τ_i, $\varphi(i)$ and γ_i are rational numbers (the first four are integers) possessing certain properties and may be chosen in a variety of different ways. Theorem 2 asserts that under the same conditions there are numbers $\alpha_1, \alpha_2, \cdots, \alpha_s$ of a similar form such that the system of functions $\alpha_1\lambda^x f(x), \alpha_2\lambda^x f(x), \cdots, \alpha_s\lambda^x f(x)$ are uniformly distributed in s-dimensional space. Theorem 3 states that under the same conditions the functions $\alpha\lambda^x f_1(x), \alpha\lambda^x f_2(x), \cdots, \alpha\lambda^x f_s(x)$ are uniformly distributed in s-dimensional space, where the polynomials $f_1(x), f_2(x), \cdots, f_s(x)$ have integral coefficients and are linearly independent. These theorems generalise results obtained by the author in earlier papers [Trudy Mat. Inst. Steklov. **38**, 87–96 (1951); Doklady Akad. Nauk SSSR (N.S.) **84**, 13–16 (1952); these Rev. **14**, 143, 144]. They depend upon the following fundamental lemma which states that for any integers $a\geqq 0$, $r\geqq 1$,

$$\left|\sum_{x=1}^{r\rho^r}\exp\left\{\frac{2\pi i\psi(x)f(a+x)}{p(\lambda^r-1)}\right\}\right|<Cr\{N_p+r+p\log(a+r\tau)\}.$$

Here p is any prime, $\psi(x)$ is a function defined recursively as a linear combination of $\psi(x-1), \psi(x-2), \cdots, \psi(x-n)$ and is periodic with period $\tau\equiv 0 \pmod p$, and N is the number of roots of the congruence $\psi(x)\equiv 0 \pmod p$ for $1\leqq x\leqq\tau$. *R. A. Rankin* (Birmingham).

Tsuji, Masatsugu. On the uniform distribution of numbers mod. 1. J. Math. Soc. Japan **4**, 313–322 (1952).

The author generalizes the concept of uniform distribution mod 1 of sequences as follows: Let $\{\lambda_n\}$ be a non-increasing sequence of positive numbers such that $\sum_1^n\lambda_n=\infty$. Let $\{x_n\}$ be a sequence of reals of fractional parts $\xi_n=x_n-[x_n]$. Let I be a subinterval of $[0, 1]$, $\varphi(x)$ its characteristic function. If for every I

$$\lim_{n\to\infty}\frac{\lambda_1\varphi(\xi_1)+\cdots+\lambda_n\varphi(\xi_n)}{\lambda_1+\cdots+\lambda_n}=\text{length of } I,$$

then we say that the sequence $\{x_n\}$ is $\{\lambda_n\}$-uniformly distributed mod 1. The (ordinary) uniform distribution mod 1 is obtained when $\lambda_n=1$ for all n. Weyl's theorem [Math. Ann. **77**, 313–352 (1916)] generalizes as follows: $\{x_n\}$ is $\{\lambda_n\}$-unif. distrib. mod 1 if and only if for $k=1, 2, \cdots$ we have $\sum_{n=1}^{n}\lambda_n\exp(2\pi kx_n i)=o(\sum_1^n\lambda_n)$. Fejér's theorem [Pólya and Szegö, Aufgaben und Lehrsätze aus der Analysis, Bd. 1, Springer, Berlin, 1925, p. 72, Problem 174] is derived and generalized by a new method using Euler's summation formula. A new generalization of van der Corput's fundamental inequality [Acta Math. **56**, 373–456 (1931), p. 407] allows one to derive the following extension of van der Corput's theorem: Let the above sequence $\{\lambda_n\}$ be also such that λ_n/λ_{n+k} is a decreasing function of n for each $k=1, 2, \cdots$. If $\{g(n+h)-g(n)\}$ is $\{\lambda_n\}$-unif. distrib. mod 1 for each $h=1, 2, \cdots$, then also $\{g(n)\}$ is $\{\lambda_n\}$-unif. distrib. mod 1. As a very special case of applications of these theorems it is found that $\{\log n\}$ is $\{1/n\}$-unif. distrib. mod 1, illustrating the smoothing effect of the sequence $\{\lambda_n\}$ in this theory. *I. J. Schoenberg.*

LeVeque, W. J. On uniform distribution modulo a subdivision. Pacific J. Math. **3**, 757–771 (1953).

Let $\Delta=(0=z_0<z_1<z_2<\cdots)$ $(z_n\to\infty)$ be a subdivision of the interval $(0, \infty)$, and let $\phi(x)$ be defined by (i) $\phi(z_n)=n$, (ii) $\phi(x)$ linear in each interval $[z_i, z_{i+1}]$. A sequence of positive numbers x_n is called u.d. (=uniformly distributed) (mod Δ) if $\phi(x_n)$ is u.d. (mod 1). The author gives a number of sufficient conditions. He proves, moreover, for some sequences containing a parameter, that they are u.d. (mod Δ) for almost all values of the parameter. We quote here: (i) If $\delta_n=z_n-z_{n-1}\downarrow 0$, $\delta_n=O(x^{-1})$, then $\{k\theta\}$ is u.d. (mod Δ) for almost all θ; (ii) if $z_n=g(n)$, $g(x)\uparrow\infty$, $\{g'(x)/g(x)\}\downarrow 0$, $\{g'(x)/g(x)\}=O(x^{-1})$, then $\{\alpha^k\}$ is u.d. (mod Δ) for almost all $\alpha>1$.

The reviewer remarks that in the proof of theorem 4 the author gives a false quotation of a principle proved in an earlier paper: the integral in (3) should read

$$\int_a^b\exp[hi(f_j(x)-f_k(x))]dx,$$

and the formula should be required for each integer h. The remainder of the proof of theorem 4 is easily revised in this respect. *N. G. de Bruijn* (Amsterdam).

Yaglom, A. M., and Yaglom, I. M. An elementary derivation of the formulas of Wallis, Leibnitz and Euler for the number π. Uspehi Matem. Nauk (N.S.) **8**, no. 5(57), 181–187 (1953). (Russian)

ANALYSIS

Ezrohi, T. G. A general form of the remainder terms of several n-dimensional approximation formulas. Dopovidi Akad. Nauk Ukrain. RSR **1952**, 174–179 (1952). (Ukrainian. Russian summary)

E. Remès [Acad. Sci. RSS Ukraine. Rec. Trav. [Zbirnik Prace] Inst. Math. **1940**, 47–82; C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 129–133 (1940); diese Rev. **2**, 195] bemerkte, dass man zahlreiche Approximationsformeln mit Restgliedern aus der Stieltjesintegraldarstellung der allgemeinen stetigen Linearform im Raume C_n herleiten kann. Verfasser baut diese Methode aus und bestimmt die Fehlerglieder in gewissen Formeln für angenäherte Integration, die von L. A. Lyusternik und V. A. Ditkin [Doklady Akad. Nauk SSSR (N.S.) **61**, 441–444 (1948); diese Rev. **10**, 153] angegeben wurden. *K. Zeller* (Philadelphia, Pa.).

Stampacchia, Guido. Approssimazione di una funzione su una superficie. Rend. Accad. Sci. Fis. Mat. Napoli (4) **19** (1952), 90–97 (1953).

Soit Σ une surface réunion d'un nombre fini de morceaux que, par un choix convenable des axes, on peut ramener à la forme: $z=z(x, y)$, z ayant des dérivées partielles premières

continues, et telle, de plus, que la distance de deux quelconques de ses points ne dépasse pas 1. L'auteur démontre la variante suivante de résultats connus, posant:

$$\Phi_n(x, y, z) = \Phi_n(P) = \frac{\pi}{n+1}\int_\Sigma \phi(Q)(1 - |PQ|^2)^n d\sigma_Q$$

où ϕ est une fonction donnée sur Σ; P un point de Σ de coordonnées x, y, z; Q un point variable de Σ, $d\sigma_Q$ l'élément d'aire en Q, $|PQ|$ la distance de P à Q; alors on a $\Phi_n(P) \to \phi(P)$ partout sur Σ, sauf peut-être au bord de Σ, si ϕ est continue; presque partout si ϕ est sommable (d'où la conclusion que $\int_\Sigma |\Phi_n|^p d\sigma$ est borné si ϕ appartient à L^p $(p>1)$). Lorsque $\phi(P)$ a des dérivées partielles premières, on a des résultats analogues lorsque $z(x, y)$ a des dérivées partielles secondes continues. Il y a une erreur sans conséquence sur la constante que l'auteur appelle k_n.

J. Favard (Paris).

*Pólya, G., und Szegö, G. **Aufgaben und Lehrsätze aus der Analysis. Erster Band. Reihen, Integralrechnung, Funktionentheorie.** 2te Aufl. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete. Bd XIX. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1954. xvi+338 pp. DM 24.00; Ganzleinen DM 27.60.

*Pólya, G., und Szegö, G. **Aufgaben und Lehrsätze aus der Analysis. Zweiter Band. Funktionentheorie, Nullstellen, Polynome, Determinanten, Zahlentheorie.** 2te Aufl. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete. Bd XX. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1954. x+407 pp. DM 28.40; Ganzleinen DM 32.00.

The first edition of these two volumes was published in 1925. There are no changes in this edition.

Karlin, S., and Shapley, L. S. Geometry of moment spaces. Mem. Amer. Math. Soc. no. 12, 93 pp. (1953).

Intéressant exposé de la géométrie des moments des distributions définies par des fonctions non décroissantes $\phi(t)$ sur un segment $[0, 1]$; les questions sont traitées avec soin et élégance. Après normalisation par $\int_0^1 d\phi = 1$, étude de l'espace D^n des moments jusqu'à l'ordre n et son espace dual P^n (celui des polynomes $\geqq 0$ dans $[0, 1]$); la frontière de ces espaces est étudiée en détail; comme fruit on trouvera un résultat nouveau sur la représentation des polynomes positifs. On définit ensuite très facilement un simplexe S^n circonscrit à D^n (et le contenant), puis, à partir de là on établit divers résultats classiques. Utilisant ensuite les formes quadratiques et les déterminants de Hankel on caractérise les faces de D^n, enfin on étudie géométriquement l'ensemble des fonctions qui admettent des moments donnés jusqu'à un ordre donné; les suites de polynomes orthogonaux se présentent d'une façon très naturelle. L'ouvrage se termine par l'exposé de certaines caractères géométriques de la suite des polynomes (trigonométriques) de Tchebycheff, et enfin par des considérations de symétrie.

J. Favard (Paris).

Corominas, Ernest, et Sunyer i Balaguer, Ferran. Sur des conditions pour qu'une fonction infiniment dérivable soit un polynome. C. R. Acad. Sci. Paris **238**, 558–559 (1954).

[In the original, "polynome" is misprinted "polymone".] The authors state several theorems, indicating that they prove them by category arguments. For example, if $f(x)$ is infinitely differentiable, and some derivative (of order depending on x) vanishes at each x, then $f(x)$ is a polynomial. "Each x" may be replaced by "each $x \in L$" if and only if the complement of L contains no perfect set. The condition that some $f^{(\nu)}(x) = 0$ may be replaced by the condition that some $f^{(\nu)}(x) \in H$, a countable set; but not if H contains a perfect set.

R. P. Boas, Jr. (Evanston, Ill.).

Rosenthal, Arthur. On functions with infinitely many derivatives. Proc. Amer. Math. Soc. **4**, 600–602 (1953).

L'auteur donne une nouvelle démonstration du théorème classique de Borel d'après lequel, quelle que soit la suite a_n, il existe une fonction $f(x)$, indéfiniment dérivable sur la droite telle que $f^{(n)}(0) = a_n$. L'auteur donne l'expression analytique de la fonction correspondante sous la forme d'une série.

S. Mandelbrojt (Paris).

Theory of Sets, Theory of Functions of Real Variables

Neumer, Walter. Zur Konstruktion von Ordnungszahlen. I. Math. Z. **58**, 391–413 (1953).

A system of notation is set up for an initial segment of the second number class, which is constructive in the sense of Church [Bull. Amer. Math. Soc. **44**, 224–232 (1938)] and Kleene [J. Symbolic Logic **3**, 150–155 (1938)]. The author intends to show in a continuation of his paper that a certain number in his system S is the limit of the totality of ordinals represented in the n-ary $(n = 3, 4, 5, \cdots)$ function systems of Ackermann [Math. Z. **53**, 403–413 (1951); these Rev. **12**, 579], and that S can also be used in connection with arbitrary ordinal numbers and normal functions.

F. Bagemihl (Princeton, N. J.).

Bagemihl, Frederick. Congruous and incongruous one-to-one correspondences. Math. Scand. **1**, 256–260 (1953).

Let Γ be a one-to-one correspondence between M and N where $M = N$ or $M \neq N$; if $\Gamma(x) \neq x$, x is called a free element relative to Γ. Let K be a disjointed system of sets and p a cardinal $\geqq 1$; then Γ is called: (a) congruous, (b) p-congruous, (c) at least p-incongruous, (d) p-congruous relative to K, according as, respectively: (a) all pairs of corresponding elements are in the same element of K, (b) Γ is congruous and each $X \in K$ contains exactly p pairs of corresponding elements, (c) (resp. (d)) if $\{X, Y\} \in K$, $X \neq Y$, then there are $\geqq p$ (resp., exactly p) elements z so that $\{z, \Gamma(z)\}$ does not belong to an element of $\{X, Y\}$. Th. 1: Let G be a class of $\aleph_\alpha$ one-to-one mappings each of which has $\geqq \aleph_\alpha$ free elements; let $2 \leqq p \leqq \aleph_\alpha$ and E be the union of domains of various $\Gamma \in G$; then there is a partition P of E into p subsets so that each $\Gamma \in G$ is at least $\aleph_\alpha$-incongruous relative to P. The paper also contains special statements of Th. 1 dealing with the continuum of real numbers and the set R of rational numbers and which are connected with some of Skolem's theorems [Skr. Vid. Kristiania. I. Mat.-Nat. Kl. **1920**, no. 4, 1–36, in particular Th. 3, p. 32]. Let P be any partition of R into $\leqq \aleph_0$-many everywhere dense sets; then between any 2 (distinct or no) intervals of R there exist $2^{\aleph_0}$ similarities which are $\aleph_0$-congruous relative to P (Th. 3).

G. Kurepa (Zagreb).

Fodor, G. Proof of a conjecture of P. Erdös. Acta Sci. Math. Szeged **14**, 219–227 (1952).

Soit E un ensemble non dénombrable de puissance $\mathfrak{m}$ et $\mathfrak{n}$ un nombre cardinal donné, tel que $\aleph_0 \leqq \mathfrak{n} < \mathfrak{m}$. Si $R \subset E \times E$ est une relation binaire entre éléments de E telle que card $R(x) < \mathfrak{n}$, R est coloriable avec moins de $\mathfrak{n}$ couleurs (c'est-à-dire, il existe une relation d'équivalence $U \subset E \times E$ telle que $R \cap U \subset \Delta$ et card $(E/U) \leqq \mathfrak{n}$). Ce théorème, qui avait été conjecturé par P. Erdös [Proc. Amer. Math. Soc. **1**, 127–141 (1950), pp. 133–137; ces Rev. **12**, 14] donne immédiatement une démonstration de la conjecture de Ruziewicz (c'est-à-dire, il existe un sous ensemble $X \subset E$ tel que $R \cap (X \times X) \subset \Delta$ et card $X = \mathfrak{n}$) dans le cas où $\mathfrak{m}$ ne peut être décomposé en une somme de $\mathfrak{n}$ ou d'un nombre moindre de nombres cardinaux dont chacun est plus petit que $\mathfrak{m}$. *J. Riguet.*

Farah, E. Sur le bon ordre de l'ensemble des puissances des parties d'un ensemble donné. Summa Brasil. Math. **3**, 37–42 (1953).

A demonstration of the (known) result that the axiom of choice implies that the set of powers of the subsets of a given set is well-ordered by the customary ordering relation.

F. Bagemihl (Princeton, N. J.).

Denjoy, Arnaud. La mesure des ensembles géométriques. C. R. Acad. Sci. Paris **238**, 753–756 (1954).

A definition of p-dimensional measure of a point set E in the Euclidean space U_n is due to Carathéodory. In the present paper the author proposes a new definition of p-dim. measure $m_p(E)$ of a point set $E \subset U_n$. A measure $m_p(E)$ is said to be Euclidean if $m_p(E)$ is invariant for every movement in U_n, otherwise it is said to be conventional. Let B denote the collection of all subsets E of U_n whose projections E_p on every p-dim. Euclidean subspace U_p of U_n, are B-measurable in U_p. If $m_p(E, U_p)$ is the B-measure of E_p in U_p, let $m_p^*(E) = \sup m_p(E, U_p)$ for all U_p in U. Then $m_p^*(E)$ is called the apparent p-dim. measure of E. Finally, if $[e_k]$ denotes any countable decomposition of E into sets $e_k \subset B$ with diam $e_k \leqq \omega$, let $M_\omega = \inf \sum_k m_p^*(e_k)$ for every $[e_k]$. Then $m_p(E) = \lim M_\omega$ as $\omega \to 0^+$, can be thought of as a p-dim. measure of E in U_n, and $0 \leqq m_p(E) \leqq +\infty$.

If a set E has $m_{q-1}(E) = +\infty$, $m_q(E) = 0$, then a measure $m_\psi(E)$ can be defined dependent on a function ψ (and thus permitting a transfinite scale of "fractional" measures between $q-1$ and q). Let (o_k) be a countable covering of E by means of connected open sets $o_k \subset U_n$, $E \subset \sum_k o_k$, with diam $o_k \leqq \omega$. Then $m_\psi(E)$ is defined by

$$m_\psi(E) = \lim \inf \sum_k \psi[m_q^*(o_k)],$$

where inf is taken for all coverings (e_k) of E and lim as $\omega \to 0^+$. Both measures $m_p(E)$ and $m_\psi(E)$ are Euclidean measures. Extensions are proposed which are conventional measures and others which concern subsets of arbitrary Riemann spaces. This remarkable definition of measure $m_p(E)$ of a point set could be traced back to the Peano definition of area of a continuous path surface [Atti Accad. Lincei. Rend. (4) **6**, 1° semestre, 54–57 (1890)].

L. Cesari (Lafayette, Ind.).

Volkmann, Bodo. Über Hausdorffsche Dimensionen von Mengen, die durch Zifferneigenschaften charakterisiert sind. II. Math. Z. **59**, 247–254 (1953).

[For part I see Math. Z. **58**, 284–287 (1953); these Rev. **14**, 1070.] The digits $0, \cdots, (g-1)$ being divided into any m sets $G_1, \cdots$ of $w_1, \cdots$ members respectively $(\sum w_\mu = g)$, the fractions whose g-adic developments digits of these sets have asymptotic frequencies ζ_μ $(\sum \zeta_\mu = 1)$ form a set of dimension $\sum \zeta_\mu \log_g (w_\mu/\zeta_\mu)$. Sharper dimensional classification is also obtained for certain sets of dimension 0.

H. D. Ursell (Leeds).

Volkmann, Bodo. Über Hausdorffsche Dimensionen von Mengen, die durch Zifferneigenschaften charakterisiert sind. III. Math. Z. **59**, 259–270 (1953).

Theorem 1 determines the dimension of the set of fractions whose g-adic development does not contain a given sequence $F \equiv f_1 \cdots f_i$ as $\log_g \gamma$, where γ is the greatest root of an algebraic equation $f(z; F, g) = 0$ of degree i. Th. 2 is a related result on linear graphs. Th. 6 compares $\gamma(F)$ with $\gamma(F')$: the result 6(c) is misstated. Th. 7 concerns the number of integers $k \leqq x$ whose g-adic development does not contain F: some changes are needed if F begins with 0.

H. D. Ursell (Leeds).

Frisancho Pineda, Ignacio. Dimensional theory of numbers. Revista Mat. Hisp.-Amer. (4) **13**, 224–228 (1953). (Spanish)

Let $A \geqq 0, x, y, \cdots, w$ be N real numbers. For the hypercomplex number $A(a_1 + ia_2 + ja_3 + \cdots + na_N)$ defined by the formulas

$$\begin{aligned} a_1 &= \cos x \cos y \cdots \cos w, \\ a_2 &= \sin x \cos y \cdots \cos w, \\ a_3 &= \sin y \cos z \cdots \cos w, \\ &\cdots \\ a_N &= \sin w, \end{aligned}$$

introduce the notation $A \exp [ix; jy; \cdots; nw] = Z_{(N)}$. If for such numbers addition is defined in the usual way and multiplication is defined by the formula

$$Z_{(N)} Z'_{(N)} = AA' \exp [i(x+x'); j(y+y'); \cdots; n(w+w')],$$

the result is a field. There are quite a few misprints, and in §3., $a_1 = a_2 = \cdots = a_N = 0$ should be replaced by $A = 0$.

T. A. Botts (Charlottesville, Va.).

Bonferroni, Carlo. Una proprietà generale delle funzioni. Boll. Un. Mat. Ital. (3) **8**, 384–390 (1953).

Let f be a function defined on an interval I of a p-dimensional Euclidean space with values in a q-dimensional Euclidean space. An interlimit of f at P_0 is a number Q_0 such that for each $\epsilon > 0$ there is a point P within ϵ of P_0 such that $f(P)$ is within ϵ of Q_0. It is proved that f is intercontinuous (that is, $f(P_0)$ is an interlimit of f at P_0) at all but a countable set of points of I. Various generalizations and corollaries are given, of which the most familiar is the theorem that if f is defined on a real interval and if the ordinary right and left hand limits exist at each point, then f is continuous at all but a countable set of points.

M. M. Day (Urbana, Ill.).

Padmavally, K. On the roots of equation $f(x) = \xi$ where $f(x)$ is real and continuous in (a, b) but monotonic in no subinterval of (a, b). Proc. Amer. Math. Soc. **4**, 839–841 (1953).

The function $f(x)$ being defined for $a \leqq x \leqq b$, with lower and upper bounds m and M in this interval, let $S(y)$ denote the set of values of x for which $f(x) = y$. It is known [see S. Minakshisundaram, J. Indian Math. Soc. (N.S.) **4**, 31–33 (1940); these Rev. **1**, 303] that, if $f(x)$ is continuous and non-differentiable, then $S(y)$ is non-enumerable for almost all y such that $m \leqq y \leqq M$. It is here proved that, if

$f(x)$ is continuous but is not monotonic in any interval, then $S(y)$ has the power of the continuum for a set of values of y which is of the second category.

U. S. Haslam-Jones (Oxford).

Kac, I. S. On the structure of singular functions of bounded variation. Uspehi Matem. Nauk (N.S.) **8**, no. 5(57), 157–159 (1953). (Russian)

Let V be the space of functions f of bounded variation on $[0, 1]$ with $f(0)=0$, let V_s be the set of singular functions in V, and let K be the subset of V_s consisting of those f for which the union of the intervals of constancy of f is of measure one. It is known [N. N. Lusin, Integral and trigonometric series, Gostehizdat, Moscow-Leningrad, 1951; these Rev. **14**, 2] that every f in V can be represented as $f=\sum_j f_j$, f_j in K, where the series is uniformly convergent on the interval $[0, 1]$. $K\subseteq V_s$ and it is well known that V_s is closed in V when distance between functions is the total variation of the difference. This note shows that K is dense in V_s in this metric. *M. M. Day* (Urbana, Ill.).

Kuipers, L. Distribution modulo 1 (functions $f(t)$ with bounded $tf'(t)$). Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 478–483 (1953).

The author continues his work on uniform distribution. Among other results the author proves the following theorem. Let F be a set of intervals Q: $0<S\leqq t\leqq T$, where S and T are continuous functions of a parameter σ ranging from σ_0 to ∞, such that $S\to\infty$, $T-S\to\infty$, $T/S\to\lambda\geqq 1$ as $\sigma\to\infty$. Let $f(t)$ be a differentiable function with $tf'(t)$ bounded $(0<t<\infty)$. Then $f(t)$ is not c-u-d (mod 1) in the intervals Q of F. [For the definition of c-u-d see Kuipers and Meulenbeld, same Proc. **52**, 1151–1157, 1158–1163 (1949); these Rev. **11**, 423.] *P. Erdös.*

Sambo, Alberto. Sulla derivazione delle funzioni composte. Rend. Accad. Sci. Fis. Mat. Napoli (4) **19** (1952), 153–156 (1953).

The author generalizes a result of G. Scorza Dragoni [Rend. Sem. Mat. Univ. Padova **20**, 462–467 (1951); these Rev. **13**, 827] to functions of more than two variables, namely he proves the following theorem: In the $(n+1)$-dimensional interval R $(a\leqq x\leqq b; a_k\leqq y_k\leqq b_k, k=1, 2, \cdots, n)$ a function $f(x, y_1, y_2, \cdots, y_n)$ is given which is measurable with respect to x and continuous with respect to $(y_1, y_2, \cdots, y_n)$. Moreover, in the interval I $(a\leqq x\leqq b)$ we have n continuous and almost everywhere differentiable functions $\alpha_k(x)$ with $a_k\leqq\alpha_k(x)\leqq b_k$ $(k=1, 2, \cdots, n)$. For almost every x of I the function $f(x, y_1, \cdots, y_n)$ is supposed to admit a partial derivative with respect to x and to be differentiable with respect to $(y_1, \cdots, y_n)$ at $(x, \alpha_1(x), \cdots, \alpha_n(x))$. Then the composite function $F(x)=f(x, \alpha_1(x), \cdots, \alpha_n(x))$ has an approximate derivative almost everywhere in I, equal to

$$f_x(x, \alpha_1(x), \cdots, \alpha_n(x))+\sum_{k=1}^{n} f_{y_k}(x, \alpha_1(x), \cdots, \alpha_n(x))\cdot\alpha_k'(x).$$

Another generalization of G. Scorza Dragoni's result to functions of the form $f(x(t), y(t))$ was given by A. Scorza Toso [ibid. **21**, 198–201 (1952); these Rev. **14**, 456].

A. Rosenthal (Lafayette, Ind.).

Parzen, Emanuel. Some conditions for uniform convergence of integrals. Proc. Amer. Math. Soc. **5**, 55–58 (1954).

Let m be a σ-finite measure on a space R, and for each t in an index set T and each positive integer n let $f_n(x, t)$ and $f(x, t)$ be functions measurable in x on R. For this situation the author introduces and studies the following two modes of uniform convergence. I. $f_n(x, t)\to f(x, t)$ almost everywhere m-uniformly in t if for every $\epsilon>0$ and every measurable set A of finite measure, as $N\to\infty$,

$$m\{x\in A: |f_n(x, t)-f(x, t)|>\epsilon \text{ for some } n>N\}\to 0$$

uniformly in t on T. II. $f_n(x, t)\to f(x, t)$ in measure m-uniformly in t if for every $\epsilon>0$ as

$$n\to\infty, \quad m\{x: |f_n(x, t)-f(x, t)|>\epsilon\}\to 0$$

uniformly in t on T. Under a certain "uniform summability" condition on the integrals $\int_R f_n(x, t)dm$ it is shown that either of I and II implies $\int_R |f_n(x, t)-f(x, t)|dm\to 0$ uniformly in t and also implies a similar uniform summability condition on the integrals $\int_R f(x, t)dm$. The author has encountered applications of these results in statistical theory.

T. A. Botts (Charlottesville, Va.).

Lane, Ralph E. The integral of a function with respect to a function. Proc. Amer. Math. Soc. **5**, 59–66 (1954).

The paper is concerned with a Stieltjes mean integral defined as the limit of $\sum_{i=1}^{n}\frac{1}{2}[u(x_i)+u(x_{i-1})][v(x_i)-v(x_{i-1})]$, where $a=x_0<x_1<\cdots<x_n=b$ is any subdivision of (a, b) and the limit is taken in terms of successive subdivisions. This integral as well as the corresponding one based on the limit as the maximum length of the subintervals of the subdivision approach zero were defined and treated by H. L. Smith [Trans. Amer. Math. Soc. **27**, 491–515 (1925)]. Further properties were developed by H. S. Kaltenborn [Tôhoku Math. J. **44**, 1–11 (1937), pp. 1–6]. The material of the present paper is essentially covered by the latter paper. *T. H. Hildebrandt* (Ann Arbor, Mich.).

Darbo, Gabriele. Sull'approssimazione dell'integrale di Lebesgue mediante somme di Riemann. Ann. Univ. Ferrara. Sez. VII. (N.S.) **2**, 13–16 (1953).

This paper proves the following theorem: If $f(x)$ is Lebesgue integrable on (a, b), then there exists a function $\theta(\delta)$ of closed subintervals δ of (a, b) such that to each δ there corresponds a point belonging to δ and such that $\sum_r f[\theta(\delta_r)]m(\delta_r)$ converges to the Lebesgue integral of $f(x)$ as the maximum length of δ_r approaches zero. [Essentially the same theorem is found in H. Hahn, Akad. Wiss. Wien, S.-B. IIa. **123**, 713–743 (1914); see also A. Denjoy, C. R. Acad. Sci. Paris **169**, 219–221 (1919).]

T. H. Hildebrandt (Ann Arbor, Mich.).

Bouligand, Georges. Sur les transformations ponctuelles conservant les aires. Gaz. Mat., Lisboa **14**, no. 56, 1–4 (1953).

Theory of Functions of Complex Variables

Čakalov, Lyubomir. On polar singular points of power series. Bŭlgar. Akad. Nauk. Izvestiya Mat. Inst. **1**, 69–82 (1953). (Bulgarian. Russian summary)

L'auteur démontre des théorèmes qui permettent de voir si une série $\sum c_n z^n$ admet des singularités polaires sur le cercle de convergence, ou de juger de leur nombre. Ainsi, si $c_n=O(1)$ (le rayon de convergence étant égal à un) les pôles sur $|z|=1$ ne peuvent être que simples, si $c_n=o(1)$ la fonction ne peut pas avoir de pôles sur $|z|=1$; d'autres théorèmes sont indiqués dont une généralization d'un théorème lacunaire de Mandelbrojt. *S. Mandelbrojt.*

Blambert, Maurice. Compléments à des théorèmes de E. Landau, M. Fekete et V. Bernstein. C. R. Acad. Sci. Paris **237**, 1207–1208 (1953).

The author announces a number of theorems on the singularities of Dirichlet series with a measurable sequence of exponents. The theorems are of the type proved by V. Bernstein [Leçons sur les progrès récents de la théorie des séries de Dirichlet, Gauthier-Villars, Paris, 1933, p. 85] connecting the density of exponents where "change of sign" of the coefficients occurs, and the maximal length of the singularity-free segment on the axis of holomorphy which is symmetric with respect to the real axis.

S. Agmon (Jerusalem).

Evgrafov, M. A. On completeness of systems of analytic functions near to $\{z^nP(z)\}$, $\{[\varphi(z)]^n\}$, and on some interpolation problems. Izvestiya Akad. Nauk SSSR. Ser. Mat. **17**, 421–460 (1953). (Russian)

Let $R=R(D)$ be the class of functions regular in a domain D. The following 3 properties of a set $\{f_n(z)\}$ ($f_n \in R$) are discussed: (A) every $f \in R$ is an uniform limit of finite linear combinations of the f_n in every closed $F \subset D$; (B) every $f \in R$ is equal to a series $\sum a_n f_n(z)$ (convergence uniform in $F \subset D$); (C) same as (B) with uniform convergence replaced by uniform summability by a matrix method of summation. Let $F(z,\zeta)=\sum_{n=0}^{\infty} f_n(z)\zeta^{-n-1}$. If the integral equation (1) $f(z)=(2\pi i)^{-1}\int_{|\zeta|=r} F(z,\zeta)g(\zeta)d\zeta$ has a solution $g(\zeta)$ for every $f \in R$ and if the series for F is uniformly convergent (or summable) in D, then term-by-term integration yields that $\{f_n(z)\} \in$ (B) (or (C)). If $f_n(z)=z^n p(z)+z^{n+1}k_n(z)$ (p, k_n regular in $|z|<R$), then $F(z,\zeta)=p(z)(\zeta-z)+K(z,\zeta)$, say. If K satisfies some conditions allowing the application of the standard theory of integral equations, then we have Theorem 1: $\{f_n(z)\} \in$ (B) in $|z|<|\alpha|$, where α is the zero of $p(z)$ closest to the origin. There is a set of functions $h_k(z)$ associated with the zeros α_k of $p(z)$ such that h_k is regular in $|z|>\alpha_k$ and such that those, and only those, functions satisfying $\int f(z)h_k(z)dz=0$ (integration along $|\alpha_k|+\epsilon$) ($k=1, 2, \cdots, m-1$) can be expanded in a series $\sum a_n f_n(z)$ in $|z|<|\alpha_m| \leq R$. Theorem 2. If D is a domain containing the origin and not containing the α_k, then $\{f_n(z)\} \in$ (C) in D. Let $\varphi(z) \in R(|z|<\rho)$, $\varphi(0)=0$, $\varphi'(0)=1$. If

$$f_n(z)=(\varphi(z))^n+z^{n+1}k_n(z)$$

(k_n regular in $|z|<\rho$), then under suitable assumptions on k_n, we have Theorem 3: $\{f_n(z)\} \in$ (B) in any D containing $z=0$ in which $\varphi(z)$ is univalent and which is the image of a circle $|w|<r$ by $\varphi^{-1}(w)$; $\{f_n(z)\}$ non-$\in$ (A), in a domain in which $\varphi(z)$ is not univalent. These results can be applied to interpolation problems in the following way. Let $\varphi(z,\zeta)$ be an entire function of z and ζ. Let S be the class of entire functions of the form $F(z)=\int \varphi(z,\zeta)h(\zeta)d\zeta$ where the integration is along the boundary C of a domain D and $h(\zeta)$ is regular outside and on C. Let $\{l_n\}$ be a set of linear functionals, $l_n(\varphi(z,\zeta))=f_n(\zeta)$. If, in D, $\{f_n\} \in$ (A), then $F \in S$, $l_n(F)=0$ (all n) implies $F \equiv 0$. If, in D, $\{f_n\} \in$ (B) and if there is a biorthogonal set $v_n(\zeta)$ such that every $f \in R(D)$ can be written as $\sum a_n f_n(z)$, $a_n=\int_C f(z)v_n(z)dz$, then for, $F \in S$, $F(z)=\sum l_n(F)\varphi_n(z)$, where $\varphi_n(z)=\int_C \varphi(z,\zeta)v_n(\zeta)d\zeta$. The choice $l_n(F)=F^{(n)}(q^n+\lambda_n)$,

$$\varphi(z,\zeta)=\sum_{n=0}^{\infty} q^{-n(n+1)/2}(z\zeta)^n/n!=\psi(z\zeta)$$

yields Theorem 4. Let ω_0 be the zero of least modulus of $\psi(t)$. If $\sum|\lambda_n q^{-n}|^2<\infty$ and $a_n=O(|q|^{-n(n+1)/2}(|\omega_0|-\epsilon)^n/n!)$ then $F(z)=\sum_{n=0}^{\infty}a_n z^n=\sum_{n=0}^{\infty}F^{(n)}(q^n+\lambda_n)\varphi_n(z)$, where the φ_n are certain polynomials. $|\omega_0|-\epsilon$ can not be replaced by $|\omega_0|+\epsilon$. Setting $l_n(F)=F^{(n)}(n+\lambda_n)$, $\varphi(z,\zeta)=e^{z\zeta}$ gives Theorem 5. If all singularities of $f(z)=\sum a_n z^{-n-1}$ are inside the domain $|ze^z|<1/e$ which contains the origin and if $\sum|\lambda_n|^2<\infty$, then $F(z)=\sum_{n=0}^{\infty}a_n z^n/n!=\sum F^{(n)}(n+\lambda_n)p_n(z)$, where the $p_n(z)$ are certain polynomials. The choice $l_n(F)=F^{(n)}((-1)^n+\lambda_n)$, $\varphi(z,\zeta)=e^{z\zeta}$ gives Theorem 6. Suppose $\sum|\lambda_n|^2<\infty$. If $F(z)$ is an entire function of order one, type $<\pi/4$, then

$$F(z)=\sum F^{(n)}((C-1)^n+\lambda_n)q_n(z),$$

where the $q_n(z)$ are polynomials. For every $k>\pi/4$ there is an entire function $G(z)$ of order one, type k such that $G^{(n)}((C-1)^n+\lambda_n)=0$. *W. H. J. Fuchs* (Ithaca, N. Y.).

Bagemihl, F., and Seidel, W. Spiral and other asymptotic paths, and paths of complete indetermination, of analytic and meromorphic functions. Proc. Nat. Acad. Sci. U. S. A. **39**, 1251–1258 (1953).

Let G denote a region of the extended plane not containing the point at infinity whose boundary consists of a finite number of mutually exclusive circles K_j ($j=1, \cdots, \nu$) some or all of which may degenerate to points. By a boundary path (b-path) is meant a simple continuous curve $z=z(t) \in G$, $0 \leq t<1$, which satisfies for some j, as $t \to 1$, $d(z(t), K_j) \to 0$. Here d refers to distance on the Riemann sphere. If $d(z(t), K_j) \not\to 0$, the path is termed a monotonic b-path. The following theorem is established: Let λ, μ denote non-negative integers with $\nu=\lambda+\mu$. If $\lambda>0$, let $\{S_n\}$ denote an arbitrary countable set of mutually exclusive b-paths in G each converging monotonically to one of the K_j ($j=1, \cdots, \lambda$) and let $\{w_n\}$ denote a sequence of complex numbers, some or all of which may be infinite. Then there exists a non-constant function f in G regular if $\lambda=\mu=1$ or if $\mu=0$ and no K_j ($j=1, \cdots, \nu$) is degenerate but meromorphic otherwise such that if $\lambda>0$, f possesses the asymptotic value w_n along S_n for each n, and if $\mu>0$, every b-path converging to one of the circles $K_{\lambda+1}, \cdots, K_{\lambda+\mu}$ is a path of complete indetermination of f (i.e. the image of the path with respect to f is everywhere dense in the plane). The proof is based upon the use of approximation methods. A number of corollaries are given. Several sufficient conditions are given that guarantee that certain b-paths be paths of complete indetermination of f. *M. Heins* (Providence, R. I.).

Havinson, S. Ya. On some nonlinear extremal problems for bounded analytic functions. Doklady Akad. Nauk SSSR (N.S.) **92**, 243–245 (1953). (Russian)

As in a previous note [same Doklady (N.S.) **88**, 957–959 (1953); these Rev. **14**, 967], G is an $n$ply connected bounded region with boundary Γ consisting of n analytic curves; B^1 consists of functions analytic in G and bounded by 1; $l_k(f)=\int_\Gamma f(x)\omega_k(x)dx$, where $\omega_k(x)$ are analytic on Γ, while on none of the contours forming Γ does any linear combination of ω_k coincide with a function representable in G by its Cauchy integral; A_r is the r-dimensional (convex) set of points $(l_1(f), \cdots, l_r(f))$. From the results of the author's previous note it follows that if $\phi(c_1, \cdots, c_r)$ is a function defined on A_r and such that $\sup|\phi|$ is attained only on the boundary of A_r, then the supremum of $|\phi(l_1(f), \cdots, l_r(f))|$ for f in B^1 is attained for a constant or for a function mapping G on an n-sheeted circle [the possibility of constant extremal functions was overlooked in the previous note]. As a consequence, the same conclusion is obtained for the extremal functions for $\sup\sum_{k=0}^{r}|c_k|^{2k}$, where c_k are the Taylor coefficients of $f(z)$ about a fixed point of G. Then

an analogous result is presented for

$$\sup \int_R \left\{ \left| \int_\Gamma f(x)\omega(x,t)dx \right|^p \right\} d\mu,$$

where the set R, measure μ and function ω satisfy suitable hypotheses. A special case of this is that if K is a collection of rectifiable arcs in G, of finite total length, and at positive distance from Γ, the greatest total length of the image of K under functions of B^1 is attained for (and only for) a mapping onto a multi-sheeted unit circle; similarly for area.

R. P. Boas, Jr. (Evanston, Ill.).

Rogosinski, W. W., and Shapiro, H. S. On certain extremum problems for analytic functions. Acta Math. **90**, 287–318 (1953).

H_p is the class of functions $f(z)$ analytic in $|z|<1$ for which

$$M_p(f)=\lim_{r\to 1}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{i\theta})|^p d\theta\right\}^{1/p}<\infty, \quad 1\leqq p<\infty,$$

and $M_\infty(f)=\sup_{|z|<1}|f(z)|$. Let $H^p\subset L^p[-\pi,\pi]$ be the set of functions ϕ which are radial limits of functions in H_p. For any $\psi \in L^q$, $1/p+1/q=1$, let I_ψ be the linear functional defined on H^p (or H_p) by: $I_\psi(\phi)=\int_{-\pi}^{\pi}\phi(t)\bar{\psi}(t)dt$. The extremal problem considered by the authors is that of determining $\sup|I_\psi(\phi)|$ taken over all $\phi \in H^p$ with $\|\phi\|_p\leqq 1$. In a previous paper Macintyre and Rogosinski [Acta Math. **82**, 275–325 (1950); these Rev. **12**, 89] discussed this in the special case in which ψ arises as the boundary value of a function $k(z)$ which is meromorphic in $|z|<1$. In the present paper, it is now observed that the extremal problem asks merely for the value of $\|I_\psi\|$. When $1<p<\infty$, L^p is reflexive and strictly convex; since H^p is a closed subspace of L^p, it too is reflexive, and its dual is in fact H^q [L. Carleson, Ark. Mat. **2**, 283–291 (1952); these Rev. **14**, 630]. It is therefore immediate that for any $\psi \in L^q$, there is a unique extremal function $\phi_0 \in H^p$ and a unique extremal kernel $\psi_0 \in H^q$ such that $I_\psi=I_{\phi_0}$ and $\|I_\psi\|=\|\psi_0\|_q=\sup|I_\psi(\phi)|=I_\psi(\phi_0)$. For $p=1$ and $p=\infty$, separate arguments show that there is always such an extremal kernal ψ_0, unique when $p=\infty$, but not necessarily so when $p=1$. Similarly, when $p=\infty$, there is always an extremal function ϕ_0, although it may not be unique; however, the familiar example

$$I(\phi)=\int_{-1}^{1}f(x)dx=-i\int_0^\pi \phi(t)e^{it}dt$$

is a functional whose norm on H^1 is not attained. If $k(z)$ is regular for $\rho<|z|<1$ and has for radial boundary values a function ψ in L^∞, then I_ψ has an extremal function ϕ_0 in H^1. The authors conclude by giving a more elegant proof for the complete solution which had been obtained in the previous paper cited above when $k(z)$ is meromorphic in $|z|<1$.

R. C. Buck (Madison, Wis.).

Tammi, Olli. On the extremal domains belonging to the coefficient a_3 of bounded schlicht functions. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **162**, 12 pp. (1953).

The author continues his study [see same Ann. no. **149** (1953); these Rev. **15**, 302] of the maximalization of the third coefficient of bounded schlicht functions

$$f(z,x)=z+\sum_{k=2}^{\infty}a_k(x)z^k, \quad |f(z,x)|<x^{-1}, \ |z|<1.$$

From the Löwner differential equation he shows that in each of the two ranges (1) $e^{-1}\leqq x\leqq 1$, (2) $0\leqq x\leqq e^{-1}$ there exists a unique (except for a rotation) extremal function corresponding to the maximum $|a_3(x)|$. These functions are exhibited together with the corresponding extremal domains. [See also Schaeffer and Spencer, Duke Math. J. **12**, 107–125 (1945); these Rev. **6**, 206.]

M. S. Robertson.

Šlionskiĭ, G. On finite sums of bounded univalent functions. Doklady Akad. Nauk SSSR (N.S.) **92**, 707–709 (1953). (Russian)

Let $f(z)=z+a_2z^2+\cdots$ be regular and univalent and $|f(z)|<M$ for $|z|<1$. The author proves that if $M\geqq 3$, the partial sums are univalent in $|z|<M/4(M-1)$ and this inequality is sharp. For $1<M<3$, the question remains open.

A. W. Goodman (Lexington, Ky.).

Li, En Pir. On the theory of univalent functions on a circular ring. Doklady Akad. Nauk SSSR (N.S.) **92**, 475–477 (1953). (Russian)

The author generalizes the classical Loewner differential equation [Math. Ann. **89**, 103–121 (1923)] to the case of a function $f(z)$ mapping the ring domain R, $m<|z|<M$, conformally onto the w-plane cut by two continuous curves ending respectively at $w=0$ and $w=\infty$, with $f(1)=1$. Application is then made to determine explicitly the extreme values of $|f(z)|$ for $f(z)$ regular and univalent in R with $f(1)=1$.

A. W. Goodman (Lexington, Ky.).

Li, En Pir. On typically real functions on a circular ring. Doklady Akad. Nauk SSSR (N.S.) **92**, 699–702 (1953). (Russian)

A function $f(z)$ is said to be typically real in a region B containing segments of the real axis if it is regular and $\operatorname{Im} f(z)\operatorname{Im} z\geqq 0$ for z in B. Let T_q denote the set of functions $f(z)=\sum_{-\infty}^{+\infty}a_nz^n$ typically real in the ring $q<|z|<1$, continuous and real on $|z|=q$, with $a_1-a_{-1}=1$. The author obtains a representation formula for functions of the class T_q in terms of a Stieltjes integral from which he then derives sharp upper bounds for $|f^{(n)}(z)|$ and $|\arg f(z)|$.

A. W. Goodman (Lexington, Ky.).

Jabotinsky, Eri. Representation of functions by matrices. Application to Faber polynomials. Proc. Amer. Math. Soc. **4**, 546–553 (1953).

Zu Funktionen $f(z)=\sum_{n=1}^{\infty}f_nz^n$, die in der Umgebung des Nullpunktes schlicht sind, gehören mittels der Entwicklungen $(f(z))^m=\sum_{-\infty}^{+\infty}f_{m,n}z^n$, m ganz rational, unendliche Matrizen $f=\|f_{m,n}\|$, $-\infty<m,n<+\infty$. Diese Matrizen stehen in einfacher Beziehung zu den entsprechenden Matrizen der Umkehrfunktion von $f(z)$, ihrer logarithmischen Ableitung und zu den Faber'schen Polynomen von $f(z)$.

A. Pfluger (Zürich).

Hayman, Walter K. La régularité des fonctions univalentes. C. R. Acad. Sci. Paris **237**, 1624–1625 (1953).

Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be analytic and univalent in $|z|<1$. Let $\limsup|a_n|n^{-1}=\gamma$. Bazilevič proved that $\gamma\leqq\frac{1}{2}e$ [Doklady Akad. Nauk SSSR. (N.S.) **65**, 253–255 (1949); these Rev. **10**, 602]. Now the author of this note announces a proof for the best possible upper bound for γ, namely $\gamma\leqq 1$, which proves the "asymptotic" Bieberbach conjecture. The result follows from the following theorem. For $f(z)=\sum_{n=0}^{\infty}a_nz^n$ analytic and univalent in $|z|<1$, set

$$M(r,f)=\sup_{0\leqq t\leqq 2\pi}|f(re^{it})|, \quad I(r,f)=\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|d\theta.$$

Then the limits

$$\alpha=\lim_{r\to 1}(1-r)^2M(r,f),\quad \beta=\lim_{r\to 1}(1-r)I(r,f),\quad \gamma=\lim_{n\to\infty}|a_n|n^{-1}$$

exist and $\alpha=2\beta=\gamma$. Only a brief sketch of the proof of this theorem is presented in this note. *G. Springer.*

Lehto, Olli. On meromorphic functions whose values lie in a given domain. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **160**, 15 pp. (1953).

Let $w=f(z)$ be meromorphic with bounded characteristic in $|z|<1$, i.e., $T(r,f)=O(1)$. Let G be a given plane domain whose boundary Γ has positive capacity, and let the values which $f(z)$ takes in $|z|<1$ lie in G and the boundary values $w=f(e^{i\theta})=\lim_{r\to 1}f(re^{i\theta})$ belong to Γ for almost all θ on $|z|=1$. It is well-known that the "counting function" $N(r,a)$ has the property that $\lim_{r\to 1}N(r,a)=N(1,a)$ is finite for every value $a\neq f(0)$. The author [in an unpublished paper] has established the relation

$$\Phi(a)+N(1,a)=g(a,f(0),G),$$

where $g(w,f(0),G)$ is the Green's function of G with pole at $f(0)$, and

$$\Phi(a)=\frac{1}{2\pi}\lim_{r\to 1}\int_0^{2\pi}g(f(re^{i\theta}),a,G)d\theta,$$

which extends a notion which O. Frostman [Medd. Lunds Univ. Mat. Sem. **3**, 1–118 (1935)] has established for bounded $f(z)$. A value a ($\neq f(0)$) is called normal or exceptional according as $\Phi(a)$ is zero or positive. The author proves the following theorems which extend results of W. Seidel [Trans. Amer. Math. Soc. **36**, 201–226 (1934)] and O. Frostman [loc. cit.]: (1) If a is an exceptional value of $f(z)$, then a is a boundary value of $f(z)$; (2) every normal value a which is a boundary value at $z=e^{i\theta}$ is in the range of $f(z)$ at $e^{i\theta}$, i.e., there exists a sequence $\{z_n\}$, $\lim z_n=e^{i\theta}$, such that $f(z_n)=a$ for all n; (3) every value not in the range of $f(z)$ is a boundary value, except when G is simply-connected and $f(z)$ is of the form $\zeta(R(z))$, where $R(z)$ is a finite Blaschke product and $\zeta(z)$ is a schlicht function which maps $|z|<1$ onto G. *A. J. Lohwater.*

Lohwater, Arthur-J. Les valeurs asymptotiques de quelques fonctions méromorphes dans le cercle-unité. C. R. Acad. Sci. Paris **237**, 16–18 (1953).

Es werden Resultate über radiale Grenzwerte von beschränkten analytischen Funktionen im Einheitskreis $|z|<1$ auf gewisse meromorphe Funktionen übertragen. Hat eine solche meromorphe Funktion $f(z)$ im Einheitskreis nur endlich viele Nullstellen und Pole und ist in einem Intervall $\alpha<\varphi<\beta$ fast überall $\lim_{r\to\infty}|f(re^{i\varphi})|=1$, so kann die Funktion dann und nur dann über den Bogen $A=\{e^{i\varphi}|\alpha<\varphi<\beta\}$ analytisch fortgesetzt werden, wenn sie darauf weder 0 noch ∞ als asymptotischen Wert besitzt. Daraus ergeben sich Konsequenzen über das Verhalten von $f(z)$ in der Nähe einer singulären Stelle, die mit Resultaten von R. Nevanlinna und W. Seidel im Falle regulärer Funktionen in Beziehung stehen. *A. Pfluger* (Zürich).

Tsuji, Masatsugu. On a direct transcendental singularity of an inverse function of a meromorphic function. J. Math. Soc. Japan **5**, 75–80 (1953).

Dans un mémoire antérieure [Tôhoku Math. J. (2) **3**, 24–38 (1951); ces Rev. **12**, 816] l'auteur a démontré une inégalité qui étend celle qui avait été utilisée par Alfhors dans la démonstration de son théorème sur le nombre des singularités transcendentes directement critiques des fonctions inverses des fonctions méromorphes d'ordre fini [Acta Acad. Aboensis **6**, no. 9 (1932)]. Soit $w(z)$ une fonction régulière dans un domaine Δ qui peut être de connexion infinie et dont la frontière Γ composée d'un ensemble dénombrable d'arcs analytiques contient au moins une courbe allant à l'infini. On suppose $w(z)\neq 0$ dans Δ, $|w(z)|<R$ dans Δ et $|w(z)|=R$ sur Γ à distance finie. On désigne par $\Delta(r)$ la portion de Δ qui appartient au cercle $|z|<r$, et par $\Delta(r_0)$ celui des domaines constituant $\Delta(r)$ qui contient un point déterminé z_0. Si $r\theta(r)$ est la somme des arcs en nombre fini interceptés par $|z|=r$ dans $\Delta(r_0)$ l'inégalité de Tsuji est la suivante. Si l'on pose avec Ahlfors et Nevanlinna

$$S(r,\Delta)=\frac{1}{r}\int\int_{\Delta(r)}\frac{|w'|^2}{(1+|w|^2)^2}dxdy,\quad w=w(z),\quad z=x+iy,$$

$$T(r,\Delta)=\int_1^r\frac{S(r,\Delta)}{r}dr,$$

on a

$$T(r,\Delta)>\text{const.}\times\exp\left(\pi\int_{r_0}^{\infty}\frac{dr}{r\theta(r)}\right),\quad r\geqq r_0,$$

quel que soit $0<\alpha<1$. L'auteur donne une nouvelle démonstration plus simple de cette inégalité qui fait appel à un théorème de Carleman [C. R. Acad. Sci. Paris **196**, 995–997 (1953)]. *G. Valiron* (Paris).

Džrbašyan, M. M. On the integral representation and uniqueness of certain classes of entire functions. Mat. Sbornik N.S. **33(75)**, 485–530 (1953). (Russian)

The author proves, in more general form, the results announced in Doklady Akad. Nauk SSSR (N.S.) **85**, 29–32 (1952); these Rev. **14**, 33. He adds two theorems giving explicit representations involving $f^{(\alpha_n)}(0)$ for entire functions of special classes for which these derivatives determine the functions uniquely. *R. P. Boas, Jr.*

Boas, R. P., Jr. Asymptotic properties of functions of exponential type. Duke Math. J. **20**, 433–448 (1953).

Eine in der obern Halbebene $y\geqq 0$ reguläre Funktion $f(z)$ vom Exponentialtypus hat das asymptotische Verhalten (A), wenn $\lim_{r\to\infty}r^{-1}\log|f(re^{i\varphi})|=c\sin\varphi$ ist für $0<\varphi<\pi$ ausser höchstens einer Menge der äussern Kapazität 0 oder für alle φ $(0<\varphi<\pi)$, wenn r von einer Menge endlicher logarithmischer Länge ausgeschlossen wird. Ist f auf der reellen Achse beschränkt, so folgt das asymptotische Verhalten (A) nach einem Satz von Ahlfors und Heins [Ann. of Math. (2) **50**, 341–346 (1949); diese Rev. **10**, 522]. Verf. gibt eine Reihe von schwächern Bedingungen an, die je das Verhalten (A) nach sich ziehen, wie z.B. dass $\int_1^{\infty}x^{-2}\log|f(\pm x)|dx$ existiert. Dieselben Bedingungen bewirken je auch die Existenz des Grenzwertes

$$\lim_{r\to\infty}r^{-1}\log M(r)\geqq 0.$$

Ist $I(R)=\int_1^R x^{-2}\log|f(x)f(-x)|dx$ lediglich nach oben beschränkt, so gilt in (A) statt des Grenzwertes nur die Beschränktheit von $r^{-1}\log|f(re^{i\varphi})|$.

Aus diesen Resultaten ergibt sich eine Reihe von Konsequenzen, von denen hier nur einige hervorgehoben werden. Ist $J(R)$ nach oben beschränkt und genügen die Nullstellen z_n der Bedingung $\sum|I(1/Z_n)|<\infty$, so ist $I(R)$ beschränkt. Ist $I(R)$ nach oben beschränkt und die Folge $\{\lambda_n\}$ so beschaffen, dass $\arg\lambda_n\to\pi/2$, $\|\lambda_m|-|\lambda_n\|\geqq|m-n|\delta$, $\delta>0$, $m\neq n$ un $\sum|\lambda_n^{-1}|$ divergent, so ist entweder

$$\limsup|\lambda_n|^{-1}\log|f(\lambda_n)|>-\infty\ \text{oder}\ f\equiv 0.$$

Für gewisse ganze Funktionen vom Exponentialtypus kann der Typus ihres Produktes nach unten abegschätzt werden. *A. Pfluger* (Zürich).

Macintyre, A. J. Interpolation series for integral functions of exponential type. Trans. Amer. Math. Soc. **76**, 1–13 (1954).

Let $F(z)$ be an entire function of exponential type; let $f(w)$ be its Laplace transform, so that $F(z)=(2\pi i)^{-1}\int_\Gamma e^{zw}f(w)dw$. Let $\{L_n\}$ be a sequence of functionals, applicable to F. A general interpolation problem is that of finding all F for which $L_n(F)=c_n$ where $\{c_n\}$ is a preassigned complex sequence; the m-point problem is the special case of this in which F is to be recovered from the values $F^{(km+b_j)}(a_j)$ where $j=1, 2, \cdots, m$, and $k=0, 1, 2, 3, \cdots$. The m-point problem has been studied by Poritsky with $b_j=0$ [Trans. Amer. Math. Soc. **34**, 274–331 (1932)] and Gontcharoff with $b_j=j-1$ [Détermination des fonctions entières par interpolation, Hermann, Paris, 1937]. In a paper by the reviewer [Trans. Amer. Math. Soc. **64**, 283–298 (1948); these Rev. **10**, 693] an approach to the general problem was used which depended upon an expansion of the kernel e^{zw}. The author of the present paper observes that stronger results may be obtained in the Poritsky and Gontcharoff problems by expanding the function $f(w)$ instead. Set $f(z, a)=\sum_0^\infty F^{(n)}(a)/z^{n+1}$ so that $f(z)=f(z, 0)$, and $e^{az}f(z)-f(z, a)$ is entire. Using the periodic aspects of L_n for these problems, the author shows that $\Delta(z)f(z)=A(z)+B(z)$ where $\Delta(z)$ and $A(z)$ are determined by the points a_j and the values $L_n(F)$, and where $B(z)$ is some entire function. Theorem: In the Poritsky and Gontcharoff problems, F is determined uniquely by the values $L_n(F)$ up to a function of the form $\sum P_k(z)e^{\beta_k z}$ where the β_k are the zeros of $\Delta(z)$. The author also discusses the behavior of the uniqueness classes for these m-point problems when m is increased, and the location of the points specialized. The author shows that most of the results of the reviewer dealing with the convergence and summability of the Abel expansion of $F(z)$ can also be obtained by the method of associated functions, using $g(z)=\sum_0^\infty F^{(n)}(n)/z^{n+1}$ and the fact that $F(z)=(2\pi i)^{-1}\int_\Gamma e^{wz}e^w g(we^w)(1+w)dw$. *R. C. Buck.*

Blambert, Maurice. Complément à un théorème de G. Pólya. C. R. Acad. Sci. Paris **237**, 1622–1624 (1953).

Let $\theta(z)=\sum(-1)^r\gamma_r z^r$ be an entire function of exponential type zero. Let $\varphi(s)$ be analytic, and define $\psi(s)=\sum\gamma_r\varphi^{(r)}(s)$. Pólya [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1927**, 187–195] has shown that every regular point of $\varphi(s)$ is regular for $\psi(s)$, and that every "accessible" singular point of $\varphi(s)$ is singular for $\psi(s)$. The author considers functions $\varphi(s)$ defined by a Dirichlet series $\sum a_n e^{-\lambda_n s}$ $(\sigma_C<\infty)$ where the λ_n have density $D>0$, and functions $\theta(z)$ whose zeros are all real and negative. For this case he extends Pólya's theorem to all singular points of $\varphi(s)$ lying on the boundary of its half-plane of regularity. Functions $\theta(z)$ whose zeros are real and positive are also considered. *J. Korevaar.*

Popova, N. V. On integrals of a certain differential equation which map a half-plane onto a region whose boundary consists of segments of straight lines. Doklady Akad. Nauk SSSR (N.S.) **91**, 727–728 (1953). (Russian)

Summary of results on the solutions of the equation

$$\frac{\partial f(z, t)}{\partial t}+P(z, t)\frac{\partial f(z, t)}{\partial z}=0$$

by means of a function of the form

$$w=c(t)\int_0^z\prod_{k=1}^m(z-a_k(t))^{\alpha_k-1}dz,$$

and the conformal representation set up between the z and w planes. For similar investigations with the circle $|z|<1$ in place of the half plane see P. P. Kufarev [Tomsk. Gos. Univ. Učenye Zapiski **1948**, no. 8, 61–72; these Rev. **11**, 21]. *A. J. Macintyre* (Aberdeen).

Drazin, M. P. The invariant circles of a bilinear transformation. Math. Gaz. **38**, 26–29 (1954).

Tsuji, Masatsugu. Theory of meromorphic functions on an open Riemann surface with null boundary. Nagoya Math. J. **6**, 137–150 (1953).

The author treats a number of topics in the theory of meromorphic functions on parabolic Riemann surfaces. Gross' theorem is extended to the considered class of functions. The valence ν of such functions is shown to have the property that the set $[\nu(a)<\sup\nu]$ has zero logarithmic capacity. It is shown that the Riemannian images of such functions are regularly exhaustible in the sense of Ahlfors. This last result is applied to the study of covering surfaces of compact Riemann surfaces of genus $p\geqq 2$. In fact, suppose C_i $(i=1, \cdots, q\leqq p)$ are q mutually disjoint retrosections of a compact Riemann surface F of genus $p(\geqq 2)$ whose union does not separate F and let $F_q^{(\infty)}$ denote the covering surface of F obtained by joining copies of $F-\bigcup_1^q C_i$ along the C_i in the standard manner. Then it is shown that $F_1^{(\infty)}$ is parabolic whereas for $q\geqq 2$, $F_q^{(\infty)}$ is hyperbolic and there exists on $F_q^{(\infty)}$ a non-constant bounded harmonic function with finite constant Dirichlet integral. *M. Heins.*

Kuramochi, Zenjiro. On covering surfaces. Osaka Math. J. **5**, 155–201 (1953).

Among the large number of topics discussed in this paper, the following are typical. (1) Connections between Fatou's theorem and the uniformization of an abstract Riemann surface. (2) Generalizations of Ohtsuka's results [Nagoya Math. J. **3**, 91–137 (1951); these Rev. **13**, 642] concerning the Dirichlet problem on Riemann surfaces. (3) R. S. Martin's topology [Trans. Amer. Math. Soc. **49**, 137–172 (1941); these Rev. **2**, 292] and his conjecture regarding the density of nonminimal ideal boundary points. (4) Riemann surfaces which belong to the class O_{HP} but do not have the Gross property.

The readibility of the paper, undoubtedly rich in results, again [cf. Kuramochi, Osaka Math. J. **3**, 123–174 (1951); these Rev. **13**, 650] suffers from an abundance of various errors. A typical sample follows (verbatim): "Theorem 5.1. Let R' be the remaining surface after a compact set of R (an abstract Riemann surface) is removed, if $R\,\bar{\in}\,O_G$ and $\in O_{HB}$ then there exists no analytic bounded functions exist." The statement that presumably is hidden in this sentence does not hold, as can immediately be seen by considering the punctured sphere. *L. Sario* (Cambridge, Mass.).

Sario, Leo. Capacity of the boundary and of a boundary component. Ann. of Math. (2) **59**, 135–144 (1954).

Ce mémoire apporte des définitions et des propriétés d'importance capitale pour les surfaces de Riemann. Certains des résultats ont été annoncés par l'auteur dans Bull. Amer. Math. Soc. **58**, 498 (1952) sous le même titre, et dans une conférence [Contributions to the theory of Riemann surfaces, Princeton, 1953, pp. 63–76; ces Rev. **15**, 209].

1. Soit R une surface de Riemann ouverte, β sa frontière idéale, $\{S\}$ la classe des fonctions $S=s+it$ analytiques sur R sauf au centre 0 d'un disque paramétrique K où elles ont la forme $\log z+\Phi(z)$ (Φ étant analytique dans K) et de

partie réelle s uniforme. Il existe dans $\{S\}$ une fonction $s_\beta+it_\beta$ telle que $\int_\beta s\,dt-\int_\beta s_\beta\,dt_\beta=D(s-s_\beta)$, D désignant l'intégrale de Dirichlet prise sur R. Cette fonction réalise évidemment le minimum μ de $\int_\beta s\,dt$. Le nombre $c_\beta=e^{-\mu/2\pi}$ est par définition la capacité de β; il coïncide avec la capacité classique quand cette dernière est définie. De plus, le minimum de $D(e^s)$ est atteint pour s_β et égal à πc_β^{-2}. L'auteur donne ensuite l'extension suivante du lemme de Schwarz: soit w une fonction analytique sur R, de module uniforme, telle que $w(0)=0$. Si l'on a $|w|\leqq c_\beta^{-1}$, alors $|w|<e^{s_\beta}$ et $\lim_{z\to 0}|w/z|<1$, sauf si $|w|=e^{s_\beta}$. Il en déduit les égalités

$$\min_{(s)}(\sup_R e^s)=\sup_R e^{s_\beta}=c_\beta^{-1}.$$

2. Rappelons qu'une composante frontière γ de R peut être définie par une suite de domaines, décroissante et d'intersection vide, dont chacun à pour frontière relative à R une courbe fermée γ_n. Une fonction sur R a une période d le long de γ si pour toute suite γ_n définissant γ elle a la période d le long de toute γ_n pour n assez grand. Parmi les fonctions $T=t+il$ de la famille $\{S\}$ telles que l a une période 2π le long de γ, et nulle le long de tout cycle ne séparant pas γ de K, il en existe une $t_\gamma+il_\gamma$ telle que $\int_\beta dl-\int_{\beta_\gamma}dl_\gamma=D(l-l_\gamma)$. Elle réalise le minimum ν de $\int_\beta dl$. Le nombre $c_\gamma=e^{-\nu/2\pi}$ est la capacité de γ. Si chacun des c_γ est nul, β est dite absolument discontinue. Supposons maintenant R planaire; alors, parmi toutes les T telles que e^T soit univalente, le minimum de $D(e^t)$ est atteint pour t_β et égal à πc_γ^{-2}.

3. L'auteur emploie les notations suivantes pour les fonctions analytiques sur R: M, non constante et de module uniforme; S, univalente; B, bornée; D, d'intégrale de Dirichlet finie. D'autre part, O_F désigne la classe des surfaces de Riemann sur lesquelles aucune fonction n'a les propriétés représentées par F, C_β désigne celle des surfaces où c_β s'annule, C_γ celle des surfaces dont la frontière est absolument discontinue. Il démontre alors qu'on a

$$C_\beta=O_{MD}=O_{MBD}=O_{MB},\quad C_\gamma=O_{SD}=O_{SB}.$$

Il rappelle que $O_{SD}=O_{SB}$ a été obtenue par Ahlfors et Beurling [Acta Math. **83**, 101–129 (1950); ces Rev. **12**, 171].

R. de Possel (Alger).

Kuroda, Tadashi. A property of some open Riemann surfaces and its application. Nagoya Math. J. **6**, 77–84 (1953).

Let G denote a non-compact domain of a Riemann surface F whose relative boundary consists of a countable set of analytic curves clustering nowhere in F. Let $\hat{G}$ denote the double of G. $\hat{G}$ is said to belong to the class O_{HB} provided that there exist no non-constant bounded harmonic functions on $\hat{G}$; G is said to belong to the class SO_{HB} provided that the only bounded harmonic function with domain G which vanishes continuously on the relative boundary is zero; G is said to belong to NO_{HB} provided that a function bounded and continuous in $\hat{G}$, harmonic in G and with vanishing normal derivative at each point of the relative boundary of G is constant. Theorem: $G\in SO_{HB}\cap NO_{HB}$ if and only if $\hat{G}\in O_{HB}$. Related theorems and applications to recent results of L. Myrberg [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **107** (1952); these Rev. **13**, 743] and L. Ullemar [Ark. Mat. **1**, 1–11 (1949); **2**, 87–97 (1952); these Rev. **11**, 343; **14**, 470] are given.

M. Heins.

Kusunoki, Yukio. Maximum principle for analytic functions on open Riemann surfaces. Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **28**, 61–66 (1953).

A Phragmèn-Lindelöf-type theorem for a region with compact relative boundary of a parabolic Riemann surface is established.

M. Heins (Providence, R. I.).

Tôki, Yukinari. On the examples in the classification of open Riemann surfaces. I. Osaka Math. J. **5**, 267–280 (1953).

It is well known that $O_G\subset O_{HP}\subset O_{HB}$ and $O_{HD}\cap O_{AB}\subset O_{AB}$, the latter inclusion being strict. The author shows that the former inclusions are strict as well, and that O_{AB} is a proper subclass of $O_{HD}\cup O_{AB}$.

Let D_μ be the region $-2^{-2\mu-1}<\log|z|<-2^{-2\mu-2}$ exclusive of its intersection α_μ with the positive real axis. For the harmonic measure $\omega_n=\omega(z,\alpha_\mu,D_\mu)$, set $k_\mu=\min\omega_n$ on the circle $\log|z|=-3\cdot 2^{-2\mu-2}$. Let R_μ be the rectangle $-2^{-2\mu}<x<-2^{-2\mu-1}$, $0<y<\pi/2^{\tau_\mu}$, and ω_β the harmonic measure $\omega(z,\beta_\mu,R_\mu)$ of the vertical sides β_μ of R_μ; set $h_\mu=\max\omega_\beta$ on the segment $x=-3\cdot 2^{-2\mu-2}$. Here the τ_μ are increasing positive integers so chosen that $h_\mu\leqq k_\mu^2$. A hyperbolic Riemann surface W with HP-removable boundary (no nonconstant positive harmonic functions on W) is constructed by taking the disk $|z|<1$ with the slits

$$-2^{-2\mu}\leqq\log|z|\leqq-2^{-2\mu-1},\quad \arg z=\nu\cdot 2\pi/2^{\tau_\mu}$$
$$(\mu=1,2,\cdots;\ \nu=1,\cdots,2^{\tau_\mu}),$$

and by identifying the edges of the slits in the same fashion as in an earlier paper [Tôki, same J. **4**, 191–201 (1952); these Rev. **14**, 864]. If the origin is removed, the remaining surface has an HB-removable, but not an HP-removable boundary. Another proof for the strictness of the inclusions $O_G\subset O_{HP}\subset O_{HB}$ was presented by the reviewer at the Conference on Functions of a Complex Variable, Ann Arbor, 1953 (Proceedings in press).

The author's example of a surface whose boundary is AB-removable but not HD-removable is, essentially, obtained from his example for $O_{HB}\subset O_{HD}$ [loc. cit.] by replacing the annulus by a disk and omitting the inner slits.

L. Sario (Cambridge, Mass.).

Myrberg, Lauri. Über die Integration der Poissonschen Gleichung auf offenen Riemannschen Flächen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **161**, 10 pp. (1953).

The existence of solutions of the Poisson equation on a non-compact Riemann surface is established in the case of an assigned C' density. The essential idea in the construction is the introduction of compensating harmonic terms to insure convergence. Application to the representation of C' vector fields on a Riemann surface.

M. Heins.

Royden, H. L. The conformal rigidity of certain subdomains on a Riemann surface. Trans. Amer. Math. Soc. **76**, 14–25 (1954).

Ce mémoire est relatif à une surface de Riemann "finie" V, c'est-à-dire, qui est bordée par un nombre fini de contours Γ (ou bien compacte). Une "différentielle quadratique de V" est une forme différentielle $d\zeta^2$ de degré un sur V qui, par rapport à un paramètre local z, a la forme $d\zeta^2=h(z)dz^2$, $h(z)$ étant analytique (sauf peut être en un nombre fini de points de V, pour lesquels $h(z)$ a un pôle) et qui de plus est réelle et non négative sur les Γ. Une "trajectoire" de $d\zeta^2$ est un arc sur lequel $d\zeta^2$ est réelle et positive. [Notions introduites par Jenkins et Spencer, Ann. of Math. (2) **53**, 4–35

(1951); ces Rev. 12, 400.] Un "domaine fendu" relatif à $d\zeta^2$ est déduit de V par suppression d'un nombre fini d'arcs de trajectoires de $d\zeta^2$.

M. Schiffer et D. C. Spencer [Functionals of finite Riemann surfaces, sous presse] ont démontré le résultat suivant: un domaine V_1 de V, tel qu'aucune représentation conforme de V_1 dans V ne soit arbitrairement voisine de l'application identique de V_1 dans V, est un "domaine fendu". L'objet du présent mémoire est de démontrer une forme de la réciproque de ce théorème: Soit $d\zeta^2$ une différentielle quadratique de V, V_1 un domaine fendu de V relatif à $d\zeta^2$ contenant en son intérieur tous les pôles multiples de $d\zeta^2$ (c'est-à-dire, des $h(z)$). Alors la transformation identique de V_1 dans V est la seule représentation conforme f de V_1 dans V qui possède les propriétés suivantes: 1) les pôles de $d\zeta^2$ sont des points fixes de f, et au voisinage d'un pôle d'ordre $k \geqq 2$, f exprimé avec un paramètre local a la forme $f(z)=z+az^k+\cdots$; 2) f a en outre une autre point fixe; 3) les accents désignant les domaines obtenus en retirant les pôles de $d\zeta^2$, il existe une déformation homotope de f, restreinte à V_1', en la transformation identique de V_1' dans V' (la métrique étant définie par $\int |d\zeta|$).

La démonstration qui procède au moyen d'un assez grand nombre de lemmes revient à une application du principe "longueur-aire" dit parfois "de Grötzsch". On introduit sur V' une triangulation qui est euclidienne dans les plans ζ. On utilise ensuite un résultat de M. Shiffman [Courant Anniversary Volume, Interscience, New York, 1948, pp. 383–394; ces Rev. 9, 303] relatif à l'unicité de la géodésique joignant deux points sur une surface-selle. Le théorème ergodique, d'après E. Hopf, intervient à plusieurs reprises. Le lemme suivant parait fondamental: Pour chacun des pôles de $d\zeta^2$, choisissons un paramètre z et retirons de V l'image d'un disque du plan z de centre 0 et de rayon ϵ. Soit V_ϵ le domaine de V ainsi obtenu. On a

$$\iint_{V_\epsilon} |f'|\,|d\zeta|^2 \geqq \iint_{V_\epsilon} |d\zeta|^2 + O(\epsilon).$$

R. de Possel (Alger).

Hervé, Michel. Sur l'itération des transformations analytiques dans le bicercle-unité. C. R. Acad. Sci. Paris **237**, 1484–1485 (1953).

Let t denote an analytic transformation $x_1=f(x, y)$, $y_1=g(x, y)$ of the bicylinder $|x|<1$, $|y|<1$ into itself, and let τ denote any transformation which is a limit of iterated transformations $t_2=t\circ t$, $t_3=t\circ t_2$, $\cdots$. If one transformation τ changes every point of the bicylinder into an interior point, every τ has this property and t has a fix-point. If t has no fix-point, there exists a real number α such that every τ has the form $x_1=e^{i\alpha}$, $y_1=\psi(x, y)$, or there exists a real number β such that every τ has the form $x_1=\varphi(x, y)$, $y_1=e^{i\beta}$. [Cf. Hervé, Ann. Sci. École Norm. Sup. (3) **68**, 125–168 (1951); these Rev. **13**, 734.] *H. Tornehave.*

***Saxer, Walter. Sur les domaines de normalité des fonctions méromorphes de plusieurs variables.** Colloque sur les fonctions de plusieurs variables, tenu à Bruxelles, 1953, pp. 125–134. Georges Thone, Liège; Masson & Cie, Paris, 1953.

The author discusses various results on domains of normality for holomorphic and meromorphic functions of several complex variables and compares some of them with results on domains of regularity for several variables. One of the theorems which the author himself obtains in the present paper is the following: Every schlicht domain in the space of two complex variables which is finite and pseudo-convex can be considered as a domain of normality for a family of holomorphic or meromorphic functions. The proof is based upon work of K. Oka on domains of regularity or meromorphy [J. Sci. Hiroshima Univ. Ser. A. **7**, 115–130 (1937); Tôhoku Math. J. **49**, 15–52 (1942); these Rev. **7**, 290].

W. T. Martin (Cambridge, Mass.).

Peschl, Ernst, und Erwe, Friedhelm. Über beschränkte Systeme von Funktionen. Math. Ann. **126**, 185–220 (1953).

Le travail est consacré à l'étude des systèmes

$$S(z)=\{f_1(z), \cdots, f_n(z)\}$$

de n fonctions holomorphes d'une variable complexe z dans le cercle C_0: $|z|<1$, avec la condition

$$|S(z)|=[\sum_1^n f_i\bar{f}_i]^{1/2} \leqq 1$$

pour $z \in C_0$, dans le but d'obtenir pour les variétés $S(C_0)$, images de C_0 dans la boule unité B_0^n, des propriétés comparables aux propriétés connues de la représentation conforme dans le cas $n=1$. L'emploi de la notation matricielle permet une écriture des automorphismes $T(\zeta)=\{W_k(\zeta_1, \cdots, \zeta_n)\}$, $1 \leqq k \leqq n$, de B_0^n formellement comparable à ceux de C_0. On a

$$W=T(\zeta)=A\Gamma(a)\frac{\zeta-a}{1-a^\times\zeta}$$

pour une automorphisme amenant le point $a=(a_1, \cdots, a_n)$ de B_0^n à l'origine; A est une matrice unitaire; on a $\Gamma(a)=(1+v(a))^{-1}aa^\times+v(a)E$, avec $v(a)=(1-|a|^2)^{1/2}$; E est la matrice unité; la colonne $a=(a_1, \cdots, a_n)$ est traitée comme une matrice; $C^\times$ designe la matrice conjuguée transposée de C. On a alors l'expression de l'invariant sous la forme:

$$d(\zeta_1, \zeta_2)=d(\zeta_3, \zeta_1)=\left|\Gamma(\zeta_1)\frac{\zeta_2-\zeta_1}{1-\zeta_1^\times\zeta_2}\right|=d(W_1, W_2).$$

Si l'on a $S(z_k)=W_0 \in B_0^n$, pour $k=0, \cdots, m$, on a la majoration pour $z \in C_0$:

$$\left|\Gamma(W_0)\frac{S(z)-W_0}{1-W_0^\times S(z)}\right| \leqq h(z), \quad h(z)=\prod_0^m \frac{z-z_\nu}{1-\bar{z}_\nu z}.$$

La recherche pour le développement

$$S(z)=a_0+a_1z+a_2z^2+\cdots$$

des conditions nécessaire et suffisantes sur les a_i pour que $|S(z)| \leqq 1$ dans C_0 s'effectue alors grâce aux notations précédentes par l'algorithme de Schur comme pour $n=1$, en considérant la suite

$$p_{\nu+1}(z)=\Gamma(a_{\nu,0})\frac{p_\nu(z)-a_{\nu,0}}{1-a^\times_{\nu,0}p_\nu(z)}\frac{1}{z}$$

où $p_\nu=a_{\nu,0}+a_{\nu,1}z+\cdots$ et $p_0=S(z)$.

La méthode permet une extension aux systèmes $S(z)$ des théorèmes de majoration connus pour les fonctions bornées dans le cercle unité. On a

$$|\Gamma_{[S(\omega)]}S'(z)| \leqq \frac{1-|S(z)|^2}{1-|z|^2},$$

$|S'(z)| \leqq [1-|z|^2]^{-1}$, ainsi que des majorations de $|S(z)|$ et de $|S'(z)|$, $[S'=f_1'\cdots f_n']$, pour $|z|=2$, qui étendent aux systèmes $S(z)$ les "Verzerrungssätze" classiques.

L'univalence de $S(z)$ est définie dans un domaine D par les deux conditions $S(z_1) \neq S(z_2)$, $z_1, z_2 \in D$ et $|S'(z)| \neq 0$ pour

$z \in D$; on obtient l'expression du rayon minimum d'univalence de la série $S(z)=a_0+a_1z+\cdots$ en fonction de $v^{-2}(a_0)|\Gamma(a_0)a_1|<1$. Une notion d'étoilement pour la variété $W=S(z)$ proposée par les auteurs de manière à conserver l'expression de la condition $\Re[zS^{\times}(z)S'(z)]>0$ se présente moins naturellement: l'image $S[|z|<r]$ est dite étoilée par rapport à l'origine si $S(0)=0$, $S(z)\neq 0$ pour $z\neq 0$ dans $|z|<r$, et l'on exige $|\theta|<\pi/2$, θ étant l'angle de la tangente à l'image d'un rayon de C_0 avec le vecteur $S(z)$.

Enfin le théorème de Julia-Carathéodory est étendu aux systèmes S pour lesquels existe une Σ: $z_k\to 1$, $z_k \in C_0$ avec $\lim S(z_k)=e_1$, $\liminf [1-|S(z_k)|][1-|z_k|]^{-1}=\alpha_0<\infty$. On a alors

$$|1-e_1^{\times}S(z)|^2[1-|S(z)|^2]^{-1}\leqq\alpha_0|1-z|^2[1-|z|^2]^{-1};$$

quand $z\to 1$ angulairement dans C_0, les quantités

$$[1-|e_1S(z)|][1-|z|]^{-1},\quad [1-e_1^{\times}S(z)][1-z]^{-1},\quad e_1^{\times}S'(z)$$

tendent vers α_0; les limites de $[S(z_k)-e_1][z_k-1]^{-1}$ constituent la variété $e_1^{\times}\lambda=\alpha_0$. La méthode explicite les systèmes $S(z)$ pour lesquels les bornes sont atteintes dans ces divers problèmes. *P. Lelong* (Lille).

*Severi, Francesco. **Quelques problèmes se rapportant aux fonctions analytiques de plusieurs variables.** Colloque sur les fonctions de plusieurs variables, tenu à Bruxelles, 1953, pp. 9–20. Georges Thone, Liège; Masson & C^ie, Paris, 1953.

The author discusses several problems related to the theory of analytic functions of several complex variables. He gives solutions where they exist but states that they do not yet present solutions in general. The first problem is concerned with a theorem for certain analytic functionals analogous in some respects to a theorem of Liouville. The theorem involves exterior differential forms. The second problem is on a theorem of meromorphic extension for functions of several real and several complex variables, analogous to E. E. Levi's theorem. As his third problem, the author considers a uniqueness theorem for analytic functions of several complex variables, and related questions. As his fourth and final problem he considers questions related to quasi-abelian functions. The paper contains references to the literature on these topics and gives indications of known results as well as the author's own ideas on and contributions to the problems. *W. T. Martin* (Cambridge, Mass.).

Bilimovitch, Anton. Sur la mesure de déflexion d'une fonction non analytique par rapport à une fonction analytique. C. R. Acad. Sci. Paris **237**, 694–695 (1953).

Let a complex-valued function, $\omega=P(x,y)+iQ(x,y)$, be given. Define the vector-valued function

$$\bar{B}=\text{grad } Q-(e_1\times e_2)\times\text{grad } P,$$

where e_1 and e_2 are the unit vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ respectively. The length of $\bar{B}$ is given by

$$\|\bar{B}\|=(P_x-Q_y)^2+(P_y+Q_x)^2$$

and so at each point gives a measure of the deviation of ω from an analytic function. Let $\bar{B}$ be decomposed into the sum of two vectors $\bar{B}_1$ and $\bar{B}_2$ which are in the direction of grad P and grad Q respectively. The cases $\bar{B}_1=0$ and $\bar{B}_2=0$ give respectively the families P and D of para-analytic functions of M. Fréchet [same C. R. **235**, 1585–1587 (1952); these Rev. **14**, 463]. *C. J. Titus* (Ann Arbor, Mich.).

Fréchet, Maurice. Les fonctions hypercomplexes à n dimensions d'une variable hypercomplexe à p dimensions. C. R. Acad. Sci. Paris **237**, 1053–1055 (1953).

In this paper the author generalizes his definition of para-analytic functions to mappings from E^p to E^n, $p\leqq n$. Let $e_1, e_2, \cdots, e_p$ and $f_1, f_2, \cdots, f_n$ be sets of mutually perpendicular vectors in E^p and E^n respectively. Let $w=\sum^p x_ie_i$ and $\phi(w)=\sum^n X_if_i$ where $X_i=X_i(x_1, x_2, \cdots, x_p)$ are real functions. Let a_{rhj} be a set of real numbers and define $f_h\cdot e_r=\sum_{j=1}^n a_{rhj}f_j$. In an obvious way one can then combine a vector in E^p and one in E^n to give a vector in E^n. The a_{rhj} are said to determine a rule R. The function Φ is said to be differentiable at the point w_0 with respect to the rule R if (1) the X_i are differentiable in the ordinary sense at w_0; (2) there exists a vector $\Phi'(w_0)$ such that $d\Phi=\Phi'(w_0)\cdot dw$ where this multiplication is according to the rule R. The function Φ is said to be para-analytic at the point w_0 if it possesses, in a neighborhood of w_0, arbitrarily many derivatives with respect to the rule R. The case $p=2$, $n=3$ is studied in detail and a complete table of canonical forms is given. *C. J. Titus* (Ann Arbor, Mich.).

Theory of Series

Hanani, Haim. On sums of series of complex numbers. Pacific J. Math. **3**, 695–709 (1953).

Let $\{c_n\}$ be a sequence of complex numbers tending to 0, $\sum|c_n|=\infty$. A number C is 'attainable', if there is a sequence $\{\epsilon_n\}$, $\epsilon_n=\pm 1$, such that $C=\sum_{n=1}^{\infty}\epsilon_nc_n$. A linet hrough the origin making an angle γ with the positive x-axis will be called an axis of divergence, if there is a subset $\{c_{n'}\}$ such that $\sum|c_{n'}|=\infty$, and $\arg c_{n'}$ has at most the two limit points γ, $\gamma+\pi$. There is at least one axis of divergence. The convergence strip of $\{c_n\}$ is defined as follows. If $\{c_n\}$ has at least two axes of divergence, then the convergence strip is the z-plane. If $\{c_n\}$ has one axis of divergence, the convergence strip consists of all lines parallel to this axis which contain attainable points of the series $\sum e^{i\alpha}\Re(c_ne^{-i\alpha})$, where the direction $e^{i\alpha}$ is perpendicular to the axis of divergence. Theorem. If $\{c_n\}$ has two axes of divergence, then every point is an attainable point. If $\{c_n\}$ has one axis of divergence and if C is any point of the convergence strip, then there are attainable points C' such that the vector $C-C'$ is arbitrarily small and has any assigned direction which is not parallel to the axis of divergence. Conditions are given under which every point is attainable. *W. H. J. Fuchs.*

Macphail, M. S. A remark on reversible matrices. Proc. Amer. Math. Soc. **5**, 120–121 (1954).

The matrix $A=(a_{nk})$ is called reversible if for each $y=\{y_n\} \in (c)$ (the set of convergent sequences) the equations $y_n=\sum_k a_{nk}x_k$ have exactly one solution $x=\{x_k\}$. In this case there exists an inverse transformation

$$x_k=c_k\cdot\lim_n y_n+\sum_n b_{kn}y_n \quad (y \in (c))$$

[Banach, Théorie des opérations linéaires, Warsaw, 1932, p. 50]. Banach further states that the c_k are bounded. This is, however, not true, as the author shows by the transformation

$$y_{2m}=\sum_{p=0}^{m}x_{2p},\quad y_{2m+1}=2^{-m}x_{2m+1}+\sum_{p=0}^{\infty}x_{2p},$$

with the inverse

$$x_{2m}=y_{2m}-y_{2m-2},\quad x_{2m+1}=2^m(y_{2m+1}-\lim_n y_n).$$

The theorem "A reversible conservative matrix A sums a divergent sequence if and only if $\sup_k (|c_k|+\sum_n |b_{kn}|)=\infty$" has been proved by Wilansky [Bull. Amer. Math. Soc. **55**, 914–916 (1949); these Rev. **11**, 243] under the assumption that the c_k are bounded. The author shows that the theorem remains true. *K. Zeller* (Philadelphia, Pa.).

Rajagopal, C. T. On Riesz summability and summability by Dirichlet's series: a further addendum and corrigendum. Amer. J. Math. **76**, 252–258 (1954).

Let $\sigma_0(x)=A_0(x)=A(x)$,

$$\text{(A)}\quad \sigma_r(x)=\frac{A_r(x)}{x^r}=\frac{r}{x^r}\int_0^x (x-u)^{r-1}A(u)\,du \quad (r>0),$$

$$\text{(B)}\quad \bar{\sigma}_r=\limsup_{x\to+\infty}\sigma_r(x),\quad \underline{\sigma}_r=\liminf_{x\to+\infty}\sigma_r(x),\quad \bar{\sigma}_\infty=\lim_{r\to\infty}\bar{\sigma}_r,\quad \underline{\sigma}_\infty=\lim_{r\to\infty}\underline{\sigma}_r,$$

$$\text{(C)}\quad F_r(t)=\frac{t^{r+1}}{\Gamma(r+1)}\int_0^\infty e^{-tu}A_r(u)\,du.$$

In a paper under the same general title the author stated Theorem *t*: If $\bar{\sigma}_\infty=\underline{\sigma}_\infty=s$, then $\bar{\sigma}_r=\underline{\sigma}_r=s$ for all sufficiently large r [Amer. J. Math. **69**, 371–378 (1947); these Rev. **9**, 26; the corresponding result for Cesàro means of series was given by J. E. Littlewood, J. London Math. Soc. **10**, 309–310 (1935)]. In a first addendum [Amer. J. Math. **69**, 851–852 (1947); these Rev. **9**, 278] the author explained that (as the present reviewer had pointed out to him) one of his results (Corollary 2 of Theorem 1) was false, as he had assumed that, if $\sigma_r(x)$ tends to a limit as $x\to+\infty$, then the integral defining $F_0(t)$ is convergent for $t>0$, whereas actually it need only be summable $(C,r-1)$ if $r>1$ [cf. L. S. Bosanquet, Proc. London Math. Soc. (3) **3**, 267–304 (1953); these Rev. **15**, 307]. In his proof of Theorem *t* he made the same false assumption. A correct proof of Theorem *t* by a different method was later given by W. B. Pennington [J. London Math. Soc. **27**, 199–206 (1952); these Rev. **13**, 738]. In the present addendum the author gives another proof of Theorem *t*. It depends on some inequalities (Lemma I) which seem to the reviewer to be unnecessarily complicated. The following argument might have been used: Suppose that $\sigma_{r_0}(x)=O(1)$. Then the function $\Phi(t)$, defined by $\Phi(t)=F_r(t)$, exists as an absolutely convergent integral for $r\geqq r_0$, $t>0$, its value being independent of r. It follows easily from (C) that

$$\underline{\sigma}_r\leqq\liminf_{t\to+0}\Phi(t)\leqq\limsup_{t\to+0}\Phi(t)\leqq\bar{\sigma}_r\quad(r\geqq r_0).$$

This supersedes Lemma I, and it is a corollary that $\Phi(+0)=s$ (Lemma II). The result follows from Lemma III (a modified version of Theorem 1 already contained implicitly in the first addendum). *L. S. Bosanquet* (London).

Braun, Julian H. Comments on summation of slowly converging series. J. Appl. Phys. **25**, 132 (1954).

It is remarked that the formula obtained by Gumowski [same J. **24**, 1068, 1330 (1953); these Rev. **15**, 26] is a version of the Euler summation formula. *R. P. Agnew.*

Gumowski, Igor. Further comments on summation of slowly converging series. J. Appl. Phys. **25**, 133 (1954).

Despite the fact that it is a standard textbook procedure to estimate $a_0+a_1+\cdots$ by evaluating $a_0+a_1+\cdots+a_n$ and then estimating $a_{n+1}+a_{n+2}+\cdots$ by the Euler summation formula, the author contends that the idea is new and original. *R. P. Agnew* (Ithaca, N. Y.).

Wheelon, Albert D. On the summation of infinite series in closed form. J. Appl. Phys. **25**, 113–118 (1954).

Identities involving series and integrals are obtained by the following idea and modifications of it. For some functions $F(x)$, a Laplace representation $F(p)=\int_0^\infty f(x)e^{-px}dx$ exists and

$$\sum_{n=0}^\infty F(n)=\int^\infty f(x)\sum_{n=0}^\infty (e^{-x})^n dx=\int_0^\infty\frac{f(x)}{1-e^{-x}}dx.$$

In some of the author's examples, the values of the series are better known than the values of the integrals. *R. P. Agnew* (Ithaca, N. Y.).

Kuerti, G. Comment on "The summation of series involving roots of transcendental equations and related applications". J. Appl. Phys. **25**, 133–134 (1954).

A critical discussion of a paper by M. R. Speigel [same J. **24**, 1103–1106 (1953); these Rev. **15**, 214]. Historical background and precise conditions are supplied where they were omitted by Speigel. *P. W. Ketchum* (Urbana, Ill.).

Delange, Hubert. Théorèmes taubériens pour les séries multiples de Dirichlet et les intégrales multiples de Laplace. Ann. Sci. Ecole Norm. Sup. (3) **70**, 51–103 (1953).

This paper generalizes the Littlewood and the Hardy-Littlewood Tauberians to multiple Dirichlet series and multiple Laplace transforms. The corresponding results for double Taylor series (in a somewhat weaker form) were established by Knopp [Math. Z. **45**, 573–589 (1939); these Rev. **1**, 51] and Durañona y Vedia [Univ. Nac. La Plata. Serie 2: Revista **4**, 291–324 (1940); these Rev. **2**, 278]. Some of the author's results were announced previously [C. R. Acad. Sci. Paris **226**, 377–379 (1948); these Rev. **9**, 425].

We confine ourselves to some typical results. Let $s(u,v)$ be a complex function defined for $u, v>0$, and let λ, μ be constants >1. Set:

$$W_s(u,v,\lambda,\mu)=\sup_{\substack{u\leqq u'\leqq\lambda u\\ v\leqq v'\leqq\lambda v}}|s(u',v')-s(u,v)|,$$

and let $w_s(\lambda,\mu)=\sup W_s$, $w_s(\lambda,\mu)=\limsup W_s$, for $u, v>0$, $u, v\to\infty$, respectively. Suppose that

$$\phi(x,y)=\int_0^\infty\int_0^\infty xye^{-ux-vy}s(u,v)\,du\,dv$$

exists for $x, y>0$. Theorem: If $w_s(\lambda,\mu)<+\infty$ and $\phi(x,y)\to S$ when $x, y\downarrow 0$, then $\limsup|s(u,v)-S|\leqq w_s(1+0,1+0)$ $(u,v\to+\infty)$, with the usual Tauberian if $w_s(1+0,1+0)=0$. Various variants and improvements of the last theorem are proved. The author then considers the Laplace-Stieltjes transform. With suitable conditions the corresponding results follow from the previous ones. In particular, one obtains Tauberians for the double Dirichlet series:

$$\text{(*)}\qquad F(x,y)=\sum\sum a_{mn}\exp(-\lambda_m x-\mu_n y).$$

Set: $p_{mn}=\sum_0^n a_{mk}$, $q_{mn}=\sum_0^m a_{kn}$ (sum on k). Theorem: Suppose a_{mn} real and

(i) $p_{mn}\geqq -K(\lambda_m-\lambda_{m-1})/\lambda_m$, $q_{mn}\geqq -K(\mu_n-\mu_{n-1})/\mu_n$;

(ii) (*) converges, is bounded for $x, y>0$, and tends to a limit S when $x, y\downarrow 0$. Then, the partial sums S_{mn} of (*) are bounded and satisfy:

$$\limsup S_{mn}=S,\quad \liminf S_{mn}\geqq S-K(2-\alpha-\beta),$$

with $\alpha=\liminf(\lambda_m/\lambda_{m+1})$ and $\beta=\liminf(\mu_n/\mu_{n+1})$. Thus, $\lim S_{mn}=S$ if, in addition, λ_m/λ_{m+1} and $\mu_n/\mu_{n+1}\to 1$. *S. Agmon* (Jerusalem).

Artémiadès, Nicolas. Sur les séries de Dirichlet à deux variables. Bull. Sci. Math. (2) **77**, 48–51 (1953).

The author considers series in the two complex variables X and Y of the form

$$(1) \qquad \sum_{m,n} a_{m,n} e^{\lambda_m X + \mu_n Y},$$

where $0<\lambda_0<\lambda_1<\cdots\to+\infty$, $0<\mu_0<\mu_1<\cdots\to+\infty$ and the coefficients $a_{m,n}$ are complex. Putting $\Re(X)=x$ and $\Re(Y)=y$, absolute convergence of (1) is studied by considering the convergence of the series

$$(2) \qquad \sum_{m,n} |a_{m,n}| e^{\lambda_m x + \mu_n y}$$

in the plane of the two real variables x and y. The series (2) is considered ordered by increasing $m+n$, although this is, of course, merely a matter of convenience.

Let $\pi_{m,n}$ be the half-plane $\lambda_m x+\mu_n y+\log|a_{m,n}|<0$. Let Δ be the set of points belonging to all but at most a finite number of these half-planes. Then, if (2) satisfies the additional condition $\limsup_{m+n\to\infty} \log(m^2+n^2)/(\lambda_m^2+\mu_n^2)^{\frac{1}{2}}=0$ (which is the analogue for two variables of the familiar one variable condition that the abscissas of ordinary and absolute convergence coincide), the domain of convergence of (2) is the region Δ. This bears a strong resemblance to a result of Hille [Ann. of Math. (2) **25**, 261–278 (1924)] on Dirichlet series (of one complex variable) with complex exponents.

The equation of the boundary of Δ is given in polar coordinates in certain special cases. It can be written $\rho=\liminf_{m+n\to\infty}[\cos\omega/L'+\sin\omega/M']^{-1}$, where

$$L'=-\log|a_{m,n}|\lambda_m^{-1} \quad \text{and} \quad M'=-\log|a_{m,n}|\mu_n^{-1}.$$

B. Lepson (Washington, D. C.).

Ascoli, Guido. Sopra un'estensione di una formula asintotica di Laplace agli integrali multipli. Rend. Sem. Mat. Univ. Padova **21**, 209–227 (1952).

Let I be a pointset in the euclidean space S_n having 0 as point of accumulation. The contingent Ω of I at 0 (in the sense of Bouligand) is said to be full (contingente pieno) provided that almost all rays r which make up Ω have the following property: There is on r a point $P\neq 0$ such that the segment $(0, P]$ is a subset of I. All sets I which are likely to appear in applications as domains of integration are seen to have, in all their points of accumulation, full contingents. This new concept is justified by the following theorem which is the main result. Assumptions: (α) Let I be a measurable set in S_n, containing the point 0 and having 0 as point of accumulation where its contingent Ω is full with $m(\Omega)=\infty$. (β) Let $\varphi(x)$ be defined and $\geqq 0$ in I and such that the supremum of $\varphi(x)$ outside of every sufficiently small neighborhood of 0 should be $<\varphi(0)$. (γ) There is a function $P_k(x)$ defined and continuous throughout S_n, positively homogeneous of degree $k>0$, and such that in Ω we have $P_k(x)<-\alpha^2|x|^k$ ($\alpha>0$), while in I

$$\lim[\varphi(x)-\varphi(0)-P_k(x)]/|x|^k=0$$

as $x\to 0$. (δ) Let $Q_h(x)$ be measurable and defined in S_n, positively homogeneous of degree $h\geqq 0$, bounded and positive on $|x|=1$. (ϵ) Let $f(x)$ be defined and measurable in I and such that $f(x)Q_h(x)$ is summable in I. Let $f(x)$ be continuous at $x=0$, $f(0)\neq 0$. Conclusion: Then

$$\int_I f(x)Q_h(x)[\varphi(x)]^\nu dx \sim f(0)[\varphi(0)]^{\nu+(h+n)/k}\cdot\nu^{-(h+n)/k} \times \int_\Omega Q_h(x)e^{P_k(x)}dx \quad (\text{as } \nu\to+\infty).$$

This is a substantial improvement, in method of derivation and result, over the existing literature of the problem, including the recent papers of L. C. Hsu [Duke Math. J. **15**, 623–632 (1948); these Rev. **10**, 246].

I. J. Schoenberg (Philadelphia, Pa.).

Ascoli, Guido. Sopra un integrale multiplo. Univ. e Politecnico Torino. Rend. Sem. Mat. **11**, 255–259 (1952).

In connection with special applications of Laplace's asymptotic formula (see preceding review) the author was led to the evaluation of integrals of the form

$$J=\int_{S_n} f_{2m}(x)\exp(-g_2(x))dx,$$

where f_{2m} and g_2 are algebraic forms of n variables of degrees as indicated by their subscripts, g_2 being positive definite. Although the evaluation is well known if preceded by an affine transformation which reduces g_2 to its canonical form, the following new expression shows the affine invariant character of J. Let $f_{2m}(x)=\sum a_{r_1r_2\cdots r_{2m}}x_{r_1}x_{r_2}\cdots x_{r_{2m}}$, $g_2(x)=\sum b_{ik}x_ix_k$, where the tensors $A=\{a_{r_1r_2\cdots r_{2m}}\}$, $B=\{b_{ik}\}$ may be assumed to be symmetric. If $B^{-1}=\{b^{ik}\}$ it is shown that

$$J=|B|^{-\frac{1}{2}}\Gamma(m+\tfrac{1}{2})\pi^{\frac{1}{2}(n-1)}\sum a_{r_1r_2r_3\cdots r_{2m}}b^{r_1r_2}b^{r_3r_4}\cdots b^{r_{2m-1}r_{2m}},$$

where r_i ranges from 1 to n. This expression shows that $J\cdot|B|^{m+\frac{1}{2}}/\pi^{\frac{1}{2}n}$ is linear in the coefficients of f_{2m} and of degree $m(n-1)$ in the coefficients of g_2 with rational coefficients.

I. J. Schoenberg (Philadelphia, Pa.).

Fourier Series and Generalizations, Integral Transforms

Mohanty, R., and Nanda, M. On the behavior of Fourier coefficients. Proc. Amer. Math. Soc. **5**, 79–84 (1954).

Let $f(t)\sim\frac{1}{2}a_0+\sum_1^\infty(a_n\cos nt+b_n\sin nt)$ and

$$\psi(t)=f(x+t)-f(x-t)-k.$$

The authors show that $a_n=O(n^{-\delta})$, $b_n=O(n^{-\delta})$, $0<\delta<1$, and $\psi(t)=o\{\log(1/t)\}^{-1}$ imply that $\{n(b_n\cos nx-a_n\sin nx)\}$ is summable $(C,1)$ to k/π. Fejér [J. Reine Angew. Math. **142**, 165–188 (1913)] had shown that $\psi(t)=o(1)$ implied the (C,r)-summability of the corresponding sequence for $r>1$.

P. Civin (Princeton, N. J.).

Morse, Marston, and Transue, William. The generalized Fréchet variation and Riesz-Young-Hausdorff type theorems. Rend. Circ. Mat. Palermo (2) **2**, 5–35 (1953).

In the present paper the authors apply and extend the concepts and results discussed in their previous joint publications [these Rev. **10**, 601, 612; **11**, 19, 185, 512; **12**, 110]. In view of the large number of definitions involved, it is not feasible to give here a detailed account of the numerous general and special results derived in this paper. Suffice it to say that the authors obtain a number of theorems and lemmas of extreme generality which yield, as very special applications, a whole array of basic theorems concerning ordinary and generalized Fourier series, including those named after Fischer, Hausdorff, Paley, Riesz, Young, and reveal in a striking fashion the logical relationships between the fundamental phenomena in this general area.

T. Radó (Columbus, Ohio).

Have

*McLachlan, N. W. **Complex variable theory and transform calculus with technical applications.** 2d ed. Cambridge, at the University Press, 1953. xi+388 pp. $10.00.

The original edition was published in 1939 under the title, Complex variable and operational calculus with technical applications [these Rev. 1, 70]. The present volume represents a substantial revision of that edition. The mathematical analysis is treated more fully and more carefully here than it was in the first edition. Material has been added on impulses, Fourier transforms, and frequency spectra. Some new applications and additional exercises have been introduced. A short table of Laplace transforms has been added, and the bibliography has been brought up to date. The book is divided into four parts entitled Theory of complex variable, Theory of transform calculus, Technical applications, and Appendices and list of references. The applications include problems in electric circuits, vibrations, aeroplane dynamics, radio and television receivers, transmission lines, electrical wave filters, design of microphones and loud speaker horns, heat conduction and absorption of moisture. *R. V. Churchill* (Ann Arbor, Mich.).

Ghizzetti, Aldo. Sopra un fondamentale teorema nella teoria della trasformazione di Laplace. Rend. Sem. Fac. Sci. Univ. Cagliari **21** (1951), 103–115 (1952).

This is a comprehensive exposition of the main results, without proofs, concerning the following question: Let $\varphi_i(t)$ $(i=1, \cdots, n)$ be summable in every finite range $0<t<T$. By Fubini's theorem their convolution

$$\Phi(t)=\varphi_1(t)*\varphi_2(t)*\cdots*\varphi_n(t)$$

is defined for almost all positive values of t and is summable in every finite range $(0, T)$. Assume the integrals $\int_0^\infty \varphi_i(t)dt$ convergent. Under what conditions does the relation (1) $\int_0^\infty \Phi(t)dt=\prod_{i=1}^n \int_0^\infty \varphi_i(t)dt$ hold? The main and final result, that the relations $\varphi(t)=O(1/t)$ as $t\to+\infty$ $(i=1, \cdots, n)$ imply the convergence of the left side of (1) and hence the validity of (1), is discussed in its historical context. [For details see Ghizzetti, Univ. e Politecnico Torino. Rend. Sem. Mat. **9**, 251–261 (1950); these Rev. **12**, 498.]

I. J. Schoenberg (Philadelphia, Pa.).

Salinas, Baltasar R. Laplace transforms of some entire functions. Gaceta Mat. (1) **5**, 157–158 (1953). (Spanish)

Agarwal, R. P. Some inversion formulae for the generalised Hankel transform. Bull. Calcutta Math. Soc. **45**, 69–73 (1953).

The generalized Hankel transformation is defined by

$$(1)\qquad f(x)=\int_0^\infty \sum_{r=0}^\infty \frac{(-1)^r(\tfrac{1}{2}xy)^{\lambda+2r}}{r!\Gamma(1+\lambda+\mu r)}g(y)(xy)^{\frac{1}{2}}dy.$$

In a former paper [Ann. Soc. Sci. Bruxelles. Sér. I. **64**, 164–168 (1950); these Rev. **12**, 605] the author obtained the inversion of this transformation in a form similar to (1). He now obtains two new inversion formulas, one of which is a double integral, and the other, a differential operator of infinite order applied to an infinite integral. *A. Erdélyi.*

Griffith, J. L. On the interpretation of certain Laplacian operator functions. J. Proc. Roy. Soc. New South Wales **87**, 51–62 (1953).

Let $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2$ be the two-dimensional Laplace operator; the formal definition of operators such as $\sum a_n\Delta^n$ on infinitely differentiable functions is obvious. The author shows that special operators of this form can be represented as integral operators, for example

$$2\pi J_0(z\Delta^{\frac{1}{2}})w(x, y)=\int_0^{2\pi} w(x-iz\cos\phi,\ y-iz\sin\phi)d\phi.$$

He gives similar interpretations of

$$\Delta^{-\frac{1}{2}}\sin(z\Delta^{\frac{1}{2}}),\ \Delta^{-\frac{1}{2}}\sinh(z\Delta^{\frac{1}{2}}),\ \cos(z\Delta^{\frac{1}{2}}),\ \cosh(z\Delta^{\frac{1}{2}}),\ I_0(z\Delta^{\frac{1}{2}}).$$

In the final section he makes some comments on the algebra of Δ. [Note that in the paper ∇ denotes $\Delta^{\frac{1}{2}}$, not the vector operator usually denoted by ∇.] *A. Erdélyi.*

Polynomials, Polynomial Approximations

Singh, S. K. On the zeros of a class of polynomials. Proc. Nat. Inst. Sci. India **19**, 601–603 (1953).

The author proves that $P(z)=a_0+a_1z+\cdots+a_nz^n$ has at least one zero outside circle $|z|=r\geqq\frac{1}{2}$ if

$$|a_0|+|a_1|+\cdots+|a_{n-1}|\leqq n|a_n|$$

and $2^{\frac{1}{2}n(n+1)}r^{n(n+1)}|a_n/a_0|<1$. His proof uses a lemma due to V. Bernstein and M. L. Cartright on the number of zeros of a regular function $F(z)$ with $\log|F(z)|$ bounded in $|z|<R$. He also proves that all the zeros of $P(z)$ lie inside the circle $|z|<2|a_{n-1}/a_n|$ if $|a_{n-1}/a_n|>\max|a_{n-k}/a_n|^{1/k}$ for $k=2, 3, \cdots, n$. Reviewer's note: The latter theorem is a special case of ex. (30,5) in M. Marden, The geometry of the zeros . . . , Math. Surveys, no. 3, Amer. Math. Soc., New York, 1949; these Rev. **11**, 101. *M. Marden.*

Mergelyan, S. N. On the rapidity of approximation of functions by polynomials on arbitrary continua. Doklady Akad. Nauk SSSR (N.S.) **91**, 1271–1274 (1953). (Russian)

The author continues his investigations on the topic indicated in the title. Let $\rho_n(f, E)$ denote the best approximation on E to $f(z)$ by polynomials of degree not exceeding n. In the present paper E is an arbitrary bounded continuum (with connected complement), and D_r is the bounded region bounded by a level line of the complement. The author obtains an inequality for $\rho_n(f, \bar{D}_r)$; and also inequalities for $\rho_n(f, E)$ under two sets of rather general but somewhat complicated hypotheses about E.

R. P. Boas, Jr. (Evanston, Ill.).

Videnskiĭ, V. S. On weighted approximation on the real axis. Doklady Akad. Nauk SSSR (N.S.) **92**, 217–220 (1953). (Russian)

The author considers the generalization of S. Bernstein's problem about weighted polynomial approximation on the real axis in which the polynomials contain only powers x^{k_n} with a given sequence $\{k_n\}$ of integers. He obtains a generalization of a theorem of Mandelbrojt [Ann. Sci. Ecole Norm. Sup. (3) **65**, 101–138 (1948); these Rev. **10**, 436], by means of approximating the weighting function by entire functions. He was, however, apparently unaware that similar generalizations had been given by Mandelbrojt [Séries adhérentes, régularisation des suites, applications, Gauthier-Villars, Paris, 1952; these Rev. **14**, 542]. *R. P. Boas, Jr.*

Natanson, G. I. On summation of series of Jacobi polynomials by a method analogous to the method of Bernštein-Rogosinski. Doklady Akad. Nauk SSSR (N.S.) **92**, 229–230 (1953). (Russian)

Let $J_n(x;\alpha,\beta)$ be the nth normed Jacobi polynomial and

$$\alpha_n=\pi/(2n+\alpha+\beta+1),\quad x_1=x\cos\alpha_n-(1-x^2)^{\frac{1}{2}}\sin\alpha_n;$$

$$x_2=x\cos\alpha_n+(1-x^2)^{\frac{1}{2}}\sin\alpha_n.$$

Theorem 1. If $(1-x)^{\alpha/2}(1+x)^{\beta/2}f(x)\in L^2(-1,1)$ and if $S_n(x)$ is the nth partial sum of the expansion of $f(x)$ in Jacobi-polynomials, then $B_n(x)=\frac{1}{2}[S_n(x_1)+S_n(x_2)]\to f(x)$ as $n\to\infty$ at every point of the Lebesgue set of $f(x)$ in $(-1,1)$. If $f(x)$ is continuous, then $B_n(x)$ tends to $f(x)$ uniformly in $\langle-1+\epsilon,1-\epsilon\rangle$. Theorem 2. If the expansion used in Theorem 1 is $f(x)\sim\sum a_nJ_n(x;\alpha,\beta)$, then for $|x|<1$

$$R_n(x)-B_n(x)=\sum_{k=0}^{n}a_kJ_k(x;\alpha,\beta)\times\cos\left((2k+\alpha+\beta+1)\alpha_n/2\right)-B_n(x)\to 0.$$

Theorem 3. If the expansion of $f(x)$ is $(C,1)$-summable, then $R_n(x)$ tends to a limit as $n\to\infty$. Theorem 1 remains true if x_j are replaced by $x_j'=x_j+O(1/n\log n)$ $(j=1,2)$. Theorem 2 remains valid with $x_j'=x_j+O(n^{-3/2-\alpha})$ in place of x_j.

W. H. J. Fuchs (Ithaca, N. Y.).

Meyer, Burnett. On the symmetries of spherical harmonics. Canadian J. Math. **6**, 135–157 (1954).

Sei G eine der 32 Kristallklassen. Es werden alle Kugelfunktionen gegebenen Grades m bestimmt, die unter einer dieser Gruppen G invariant bleiben. Zu diesem Zweck wird zuerst mittels einer Methode von Molien die Anzahl g_m der Elemente einer Basis aller homogenen Polynome, die unter G invariant sind, bestimmt und mittels $g(t)=\sum_0^\infty g_mt^m$ der Gruppe ein erzeugendes Polynom zugeordnet. Diese Polynome werden für die einzelnen der 32 Klassen angegeben. Sei sodann h_m die Anzahl der Elemente in einer harmonischen Basis vom Grade m, d.h. einem System von linear unabhängigen Kugelfunktionen $m$ten Grades, invariant unter G. Dann gilt mit $h(t)=\sum_0^\infty h_mt^m$ dass $h(t)=(1-t^2)g(t)$. Für jede Gruppe wird das zugehörige $h(t)$ angegeben. Sind sodann $Q_i(x,y,z)$, $i=1,\cdots,h_m$, unter G invariante homogene Polynome $m$ten Grades, die mod $(x^2+y^2+z^2)$ linear unabhängig sind, dann bilden die h_m Kugelfunktionen $r^{2m+1}Q_i(\partial/\partial x,\partial/\partial y,\partial/\partial z)r^{-1}$ das Verlangte. Sie werden wiederum für die einzelnen Gruppen in Tabellen angegeben.

J. J. Burckhardt (Zürich).

Special Functions

van der Pol, B. Note on the gamma function. Canadian J. Math. **6**, 18–22 (1954).

The author proves the formula

$$\Gamma(z+1)=(z+1)^ze^{-z}\prod_{k=1}^{\infty}\left\{\left(1+\frac{1}{k+z}\right)^{k+z}\left(1+\frac{1}{k}\right)^{-k}\right\}$$

which he derives from Weierstrass' product for the gamma function. He gives a discussion of this formula, comparing it with Stirling's formula, showing directly that it satisfies the functional equation of the gamma function, and also giving some variants.

A. Erdélyi (Pasadena, Calif.).

Mikolás, Miklós. Über die Beziehung zwischen der Gammafunktion und den trigonometrischen Funktionen. Acta Math. Acad. Sci. Hungar. **4**, 143–157 (1953). (Russian summary)

The principal result of this paper is the integral representation

$$\psi(z)=-\gamma-\frac{1}{2z}-\frac{\pi}{2}\cot(\pi z)-\frac{\pi}{2}\int_0^1\tan\left(\frac{\pi t}{2}\right)\left(\frac{\sin\pi zt}{\sin\pi z}-t\right)dt$$

for the logarithmic derivative of the gamma function. This integral representation is valid for all non-integer values of z, and the integral is a (proper) Riemann integral. This result is applied to obtain integral representations of $\log\Gamma(z)$, of Euler's constant γ, of $\zeta(n+1,a)$, $\zeta(n+1)$; and also to obtain numerous relations involving $\zeta(n)$.

A. Erdélyi.

Shenton, L. R. The continued fraction for $F(a,1;c;t)$. Math. Gaz. **38**, 39–40 (1954).

The author starts with Gauss' continued fraction for $F(a,1;c;t)$, investigates the differential equations satisfied by the numerators and denominators of the convergents, shows that the "remainder" (the difference between F and its convergents) is the ratio of two hypergeometric series, the formulas being different for convergents of even and odd orders; and derives formulas which are closely related to the Wronskian of two solutions of the hypergeometric equations.

A. Erdélyi (Pasadena, Calif.).

Chakrabarty, N. K. On a generalisation of Bateman's k-function. Bull. Calcutta Math. Soc. **45**, 1–7 (1953).

The function discussed in this paper is Whittaker's well-known confluent hypergeometric function (disguised by an unconventional notation); neither the methods nor the results seem to be new.

A. Erdélyi (Pasadena, Calif.).

Fenyő, István. On an integral equation connected with functions on a higher order spherical surface. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **3**, 513–520 (1953). (Hungarian)

Let P and Q be points on the surface of the unit sphere Σ in n-dimensional Euclidean space, and let γ be the angle POQ. The characteristic functions of the integral equation

$$\phi(P)=\lambda\int_\Sigma K(\cos\gamma)\phi(Q)d\Sigma_Q$$

are spherical surface harmonics. For $n=3$ and continuous K this was proved by Funk, for any n and continuous K by Hecke, and for more general K by the reviewer [see, e.g., Erdélyi, et al., Higher transcendental functions, vol. 2, McGraw-Hill, New York, 1953, sec. 11.4; these Rev. **15**, 419]. The author now gives another proof, and an application to random walks on the surface of the hypersphere.

A. Erdélyi (Pasadena, Calif.).

Delerue, P. Sur le calcul symbolique à n variables et les fonctions hyperbesséliennes. II. Fonctions hyperbesséliennes. Ann. Soc. Sci. Bruxelles. Sér. I. **67**, 229–274 (1953).

[For part I see same Ann. **67**, 83–104 (1953); these Rev. **14**, 1082.] The hyper-Bessel function of order n with indices $a_1,\cdots,a_n$ is defined as

$$J^{(n)}_{a_1,\cdots,a_n}(x)=\sum_{r=0}^{\infty}\frac{(-1)^r[x/(n+1)]^{a_1+\cdots+a_n+r(n+1)}}{r!\Gamma(a_1+r+1)\cdots\Gamma(a_n+r+1)}.$$

This paper contains a great many relations involving these

functions, and does not lend itself to a brief summary. [In view of the review of Part I, it is only fair to the author to state that the present Part II gives a few references ranging from a paper by P. Humbert in 1936 [same Ann. 56, 26–43 (1936)] to the recent book by Voelker and Doetsch [Die zweidimensionale Laplace-Transformation, Birkhäuser, Basel, 1950; these Rev. 12, 699].] *A. Erdélyi.*

Parodi, Maurice. Sur une propriété des fonctions de Bessel. C. R. Acad. Sci. Paris **238**, 195–196 (1954).

Finite sums of integrals of the form $\int_0^\pi aJ_1[bz\sin\theta]d\theta$, a and b constants, are evaluated. *N. D. Kazarinoff.*

Marx, Imanuel. Recurrence relations for prolate spheroidal wave functions. J. Math. Physics **32**, 269–275 (1954).

This paper starts from an observation by E. T. Whittaker, that in the two-dimensional wave equation the first derivatives of solutions are also solutions. This is applied to the radial and to the angular wave functions connected with oblate and prolate spheroids. Arbitrary solutions of the spheroidal two-dimensional wave equations may be expressed by a double series of simple wave function products. This is also true for the derivatives of such solutions. From this, recurrence relations for the radial and for the angular wave functions are obtained, the coefficients of which are definite integrals of such functions and of their products. No evaluation of these integrals is given. Therefore, the general form rather than the exact equations of the recurrence relations is obtained. *M. J. O. Strutt* (Zürich).

Rushforth, J. M. A generalisation of Jacobi's fundamental formulae. Proc. Edinburgh Math. Soc. (2) **9**, 17–19 (1953).

Define the $2N$ variables X_i as linear functions of the $2N$ variables x_j by the set of equations $N(X_i+x_i)=\sum_{j=1}^{2N}x_j$ $(i=1, 2, \cdots, 2N)$. Then

$$(*)\quad N\prod_{j=1}^{2N}\vartheta_3(X_j)=\sum_{r,s=0}^{N-1}\sum\exp\left\{\frac{-2\pi irs}{N}\right\}\prod_{j=1}^{2N}\times\exp\left\{\frac{2ir}{N}\left(x_j+\frac{\pi s}{N}+\frac{\pi\tau}{N}\right)\right\}q^{-r^2/N^2}\vartheta_3\left(x_j+\frac{\pi s}{N}+\frac{\pi\tau}{N}\right).$$

Three similar formulae are obtained from (*), by increasing all variables by half periods. For $N=2$, (*) reduces to Jacobi's [Ges. Werke, Bd. I, Reimer, Berlin, 1881, pp. 502–506] formula:

$$2\vartheta_3(w)\vartheta_3(x)\vartheta_3(y)\vartheta_3(z)=\vartheta_3(w')\vartheta_3(x')\vartheta_3(y')\vartheta_3(z')+\vartheta_4(w')\vartheta_4(x')\vartheta_4(y')\vartheta_4(z')+\vartheta_2(w')\vartheta_2(x')\vartheta_2(y')\vartheta_2(z')-\vartheta_1(w')\vartheta_1(x')\vartheta_1(y')\vartheta_1(z').$$

(*) is proven algebraically, by a generalization of H. J. S. Smith's [Proc. London Math. Soc. 1, May 21 (1866), pp. 1–14] method. *E. Grosswald* (Philadelphia, Pa.).

Harmonic Functions, Potential Theory

Arsove, Maynard G. Functions representable as differences of subharmonic functions. Trans. Amer. Math. Soc. **75**, 327–365 (1953).

Une fonction $w(z)$ est dite δ-sousharmonique (presque δ-sousharmonique) dans un domaine plan Ω si $w=u-v$ (resp. $w=u-v$ presque partout), u et v étant sousharmoniques dans Ω. La fonction w admet une infinité de représentations (u, v); une telle représentation est dite canonique si les mesures négatives μ et ν associées à u et v par le théorème de décomposition de F. Riesz satisfont à $(\mu-\nu)^+=-\nu$, $(\mu-\nu)^-=-\mu$; $\mu-\nu$ sera appelée mesure associée à w. Parmi les nombreuses questions traitées par l'auteur, concernant ces fonctions, signalons: (1) l'étude des représentations canoniques, qui sont définies à une fonction harmonique près; (2) les critères de (presque) δ-sousharmonicité, qui sont d'ailleurs des conséquences d'un critère de L. Schwartz [Théorie des distributions, t. I, II, Hermann, Paris, 1950, 1951; ces Rev. 12, 31, 833] (la "distribution" Δw est une mesure), mais qui sont présentés d'une façon élémentaire, par exemple en utilisant la variation de N. Wiener [Acta Litt. Sci. Szeged 3, 7–16 (1927)] et des résultats analogues à ceux de P. C. Rosenbloom [Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, Johan Grundt Tanum, Oslo, 1952, pp. 130–138; ces Rev. 15, 220]; (3) l'étude des familles de fonctions δ-sousharmoniques: par exemple on donne des conditions suffisantes assez larges pour que le produit de deux telles fonctions soit δ-sousharmonique (ce qui n'a pas toujours lieu); en particulier: l'ensemble des différences de deux fonctions sousharmoniques bornées sur tout compact de Ω constitue une algèbre; (4) l'étude de la fonction caractéristique $T_r(w, z)$ de la fonction δ-sousharmonique w; c'est par définition la valeur moyenne sur la circonférence $C(z, r)$ de la fonction sousharmonique $\sup(u, v)$, où (u, v) est une représentation canonique quelconque de w (pour $z\in\Omega$, $T_r(w, z)$ est donc définie à une constante près pour les petites valeurs de r. On compare cette définition à celle de Privaloff [Les fonctions sousharmoniques, Glavnaya Redakciya Tehn.-Teor. Lit., Moscow-Leningrad, 1937], qui est directement calquée sur celle de Nevanlinna (correspondant au cas où $w=\log|f(z)|$, avec $f(z)$ méromorphe), et fait intervenir explicitement la mesure associée à w: il y a identité (à une constante près en r) en tout point z pour lequel la fonction de Privaloff est finie (ce qui n'a pas lieu nécessairement pour tout $z\in\Omega$); on donne des applications à l'étude locale des fonctions δ-sousharmoniques.

J. Deny (Strasbourg).

Yûjôbô, Zuiman. A theorem concerning subharmonic functions defined in a strip domain. Comment. Math. Univ. St. Paul. 1, 1–4 (1952).

This note contains a proof of the following generalization of a result due to Hardy, Ingham, and Pólya [Proc. London Math. Soc. (2) **27**, 401–409 (1928)]. If $\Psi(z)$ is continuous for $\alpha\leqq x\leqq\beta$, subharmonic for $\alpha<x<\beta$, and satisfies $-\infty<\Psi(z)\leqq ce^{k|y|}$, where c, k are constants, $c>0$, $0<k<\pi/(\alpha-\beta)$, and if $J(x)\equiv\int_{-\infty}^{\infty}\Psi(x+iy)dy$ is convergent for $x=\alpha$, $x=\beta$, then $J(x)$ exists for $\alpha\leqq x\leqq\beta$, and either $J(x)\equiv-\infty$ $(\alpha<x<\beta)$ or $J(x)$ is a convex function of $(\alpha\leqq x\leqq\beta)$. As a corollary, the author finds that if $\Psi(z)$ is (real and) continuous for $\alpha\leqq x\leqq\beta$, harmonic for $\alpha<x<\beta$, and satisfies $|\Psi(z)|<ce^{k|y|}$, and if $J(x)$ converges for $x=\alpha$, $x=\beta$, then $J(x)$ is a linear function for $\alpha\leqq x\leqq\beta$.

M. O. Reade (Ann Arbor, Mich.).

Allen, A. C., and Kerr, E. Harmonic functions and Tauberian theorems. I. J. London Math. Soc. **29**, 104–115 (1954).

For $\eta>0$, let

$$H(\xi,\eta)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\eta\,dg(t)}{\xi^2+(\eta-t)^2},$$

where $g(t)$ is a non-decreasing function with $g(0)=0$ and

$\int_{-\infty}^{\infty}(1+t^2)^{-1}dg(t)<\infty$. Loomis [Trans. Amer. Math. Soc. 53, 239–250 (1943); these Rev. 4, 199] proved that if $H(0,\eta)\to A$ as $\eta\searrow 0$, then the symmetric derivative of $g(t)$ at $t=0$ exists and is equal to A. In a note which was withdrawn because of an error, Verblunsky [J. London Math. Soc. 22, 210–216 (1948); these Rev. 9, 425] stated the following generalization of this result: if there exists a θ in $(0,\pi/2)$ such that $H(r\cos\theta,\ r\sin\theta)\to A$ as $r\searrow 0$, then $[(\pi-\theta)g(t)-\theta g(-t)]/t\to A\pi$ as $t\searrow 0$. The present authors construct an example which shows that the proposition in question is false, together with its converse. They also prove that the conclusion in Verblunsky's statement should read as follows: there exists a unique non-decreasing function $f(t)$ with $f(0)=0$ and such that

$$\int_0^{\infty}\frac{\eta\,df(t)}{\eta^2+(\xi-t)^2}=\int_0^{\infty}\frac{\eta\,d\{-g(-t)\}}{\eta^2+(\xi+t)^2}$$

for every point $\xi=r\cos\theta$, $\eta=r\sin\theta$ $(r>0)$; and

$$[g(t)+f(t)]/t\to A\pi/(\pi-\theta)$$

as $t\searrow 0$. *G. Piranian* (Ann Arbor, Mich.).

Brelot, M. La théorie moderne du potentiel. Ann. Inst. Fourier Grenoble 4 (1952), 113–140 (1954).
Expository paper.

Differential Equations

Babkin, B. N. On approximate solution by Čaplygin's method of ordinary differential equations of the first kind which are not solved for the derivative. Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 634–638 (1953). (Russian)
Čaplygin's theory of differential inequalities was formulated for differential equations explicitly solved for the derivative. This article treats the implicit case where the differential equation is in the form $F(x,y,y')=0$. Among the results obtained the following is typical: If two curves $y=u(x)$ and $y=v(x)$ issuing from the point (x_0,y_0) are such that $F(x,u,u')<0$, $F(x,v,v')>0$, and $\partial F/\partial y'>0$ in the region considered, then $u(x)<y(x)<v(x)$, where y is the solution through (x_0,y_0) and $x_0<x$. The result is used to establish a sequence $v_1(x),v_2(x),\cdots,v_n(x),\cdots$ converging to $y(x)$. *W. E. Milne* (Corvallis, Ore.).

Hartman, Philip, and Wintner, Aurel. On non-oscillatory linear differential equations. Amer. J. Math. 75, 717–730 (1953)
The authors study the asymptotic behavior of solutions of a differential equation (1) $x''+f(t)x=0$, when $f(t)$ is continuous for t large. A typical theorem is this. If $\int^{\infty}tf(t)dt$ converges and if $\int^{\infty}t^{p-1}|f(t)|^p dt<\infty$, where p is some number on the range $1\leq p\leq 2$, there then exist solutions $x(t)$ and $y(t)$ of (1) with the properties $x(t)\to 1$, $y(t)\sim t$, $x'(t)=o(t^{-1})$, $y'(t)\to 1$. *W. Leighton.*

Hartman, Philip, and Wintner, Aurel. On non-oscillatory linear differential equations with monotone coefficients. Amer. J. Math. 76, 207–219 (1954).
The authors consider the equation $u''+q(t)u=0$, where $q(t)$ is taken to be monotone and continuous. They show that a number of results concerning the qualitative behavior of the solutions can be determined from the combination of monotonicity and positivity or negativity of $q(t)$.
R. Bellman (Santa Monica, Calif.).

Hartman, Philip, and Wintner, Aurel. Linear differential equations with completely monotone solutions. Amer. J. Math. 76, 199–206 (1954).
In an earlier paper, the authors showed that $u''-f(t)u=0$ possesses a completely monotone solution over $0\leq t<\infty$, if $f(t)$ is completely monotone over the same range. This result is now generalized to nth order equations as follows: Let $-f_0''$, $-f_1'$, and $f_2, f_3, \cdots, f_n$, be completely monotone and $f_0>0$. Then the equation $f_0u^{(n)}+\sum_{k=1}^{n}(-1)^{k-1}f_ku^{(n-k)}=0$ possesses a completely monotone solution. *R. Bellman.*

Evans, Robert L. Asymptotic and convergent factorial series in the solution of linear ordinary differential equations. Proc. Amer. Math. Soc. 5, 89–92 (1954).
Considered is the differential equation $\sum_{\beta=0}^{n}P_\beta(x)y^{(n-\beta)}=0$, where the series $P_\beta(x)=\sum_{\mu=-n(\beta)}^{\infty}p_{\beta,\mu}x^{-\mu}$ ($p_{\beta,-n(\beta)}\neq 0$ and $\beta=0,\cdots,n$) converge for $|x|>x_0$; $x=\infty$ is generally a singular point. The equation has n linearly independent formal solutions, given by E. Fabry, $\sigma_\nu(x)$ $(\nu=1,\cdots,n)$, which involve certain possibly divergent series. A transformation $x=z^m$, with m suitably chosen, removes fractional powers from the n formal solutions. Using the Puiseux diagrams and Laplace integrals, consideration of the $\sigma_\nu(x)$ leads to a full set of actual solutions, involving convergent factorial series. *W. J. Trjitsinsky* (Urbana, Ill.).

Amerio, Luigi. Extension of the notion of "saddle point" to systems of two differential equations in three variables. Comm. Pure Appl. Math. 6, 435–454 (1953).
The author considers the system of equations, $x'=f(x,y,t)$, $y'=g(x,y,t)$, and shows that, under suitable assumptions on f and g, the concept of saddle-point, which is of such value in the case where f and g are independent of t, can be extended to the more general case. *R. Bellman.*

Bautin, N. N. On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type. Amer. Math. Soc. Translation no. 100, 19 pp. (1954).
Translated from Mat. Sbornik N.S. 30(72), 181–196 (1952); these Rev. 13, 652.

Pliss, V. A. A qualitative picture of the integral curves in the large and the construction with arbitrary accuracy of the region of stability of a certain system of two differential equations. Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 541–554 (1953). (Russian)
The following system has been investigated by Erugin [same journal 16, 620–628 (1952); these Rev. 14, 376]:

$$\dot{x}=f(x)+a'y,\quad \dot{y}=b'x+c'y,$$

$x(f(x)+c'x)<0$, $x(c'f(x)-a'b'x)>0$, for $x\neq 0$; $f(0)=0$. Here a', b', c' are constants and $f(x)$ is continuous. He showed that the origin is asymptotically stable, except possibly when $c'^2+a'b'=0$. In that case Erugin also showed [ibid. 14, 459–512 (1950); these Rev. 12, 412] that the transformation $y\to a'y-c'x$ reduced the system to the form

$$(2)\qquad \begin{cases}\dot{x}=y-\varphi(x), & \dot{y}=c\varphi(x)\\ c<0; & \varphi(0)=0; \quad x\varphi(x)>0 \text{ for } x\neq 0.\end{cases}$$

The exceptional case arises only if

$$\lim\int_0^x\varphi(x)dx<\infty$$

as $x\to+\infty$ or else as $x\to-\infty$. The author assumes this

situation and in addition that

$$\int_0^{+\infty}\varphi(x)dx=D<+\infty,\quad \int_0^{+\infty}\varphi(x)dx\leq\int_0^{-\infty}\varphi(x)dx,$$

$$\limsup \varphi(x)=a<+\infty \quad \text{as} \quad x\to+\infty.$$

Under these conditions he determines the boundary of the region of asymptotic stability in the cases when this boundary exists. This boundary consists necessarily of trajectories (Erugin). Incidentally he obtains the phase-portrait of the trajectories outside the region of stability.

S. Lefschetz (Princeton, N. J.).

Zaremba, S. K. Divergence of vector fields and differential equations. Amer. J. Math. 76, 220–234 (1954).

Let $x=x(t)$, $y=y(t)$ be an integral curve of the system of equations (1): $dx/X(x,y)=dy/Y(x,y)=dt$; and let $x(t_i)=x_i$, $y(t_i)=y_i$, $i=1, 2$. Let T_i be the orthogonal trajectory of the integral curves passing through the point (x_i, y_i); and let u_i be the parameter on T_i which vanishes at (x_i, y_i) and satisfies the equations

$$-dx/Y(x,y)=dy/X(x,y)=du_i/(X^2+Y^2).$$

It is shown that du_2/du_1 is given, for $u_1=0$, by the formula

$$du_2/du_1=\exp\int_{t_1}^{t_2}[X_x(x(t),y(t))+Y_y(x(t),y(t))]dt.$$

This result is used to obtain simple proofs of various known theorems concerning the integral curves of the system (1); in particular, theorems concerning the existence and stability of limit cycles. A complicated new theorem is given which is a partial converse of Bendixson's theorem asserting the non-existence of closed integral curves in a simply connected domain in which $X_x(x,y)+Y_y(x,y)$ has a constant sign.

L. A. MacColl (New York, N. Y.).

Volk, V. Ya. On inversion formulas for a differential equation with a singularity at $x=0$. Uspehi Matem. Nauk (N.S.) 8, no. 4(56), 141–151 (1953).

Let $y''+[\lambda-q(x)-(p^2-\frac{1}{4})/x^2]y=0$ on $0\leq x<\infty$, where q is continuous and p constant. Any bounded solution $\phi(x,\lambda)$ may be represented by

$$\phi(x,\lambda)=x^{\frac{1}{2}}\lambda^{\frac{1}{4}}J_p(x\lambda^{\frac{1}{2}})+\int_0^x K(x,t)t^{\frac{1}{2}}\lambda^{\frac{1}{4}}J_p(t\lambda^{\frac{1}{2}})dt,$$

where $K(x,t)$ is continuous and J_p is the Bessel function. For the case $p=\frac{1}{2}$ the analogous result was given by Povzner [Mat. Sbornik N.S. 23(65), 3–52 (1948); these Rev. 10, 299].

N. Levinson (Cambridge, Mass.).

Putnam, C. R. On the gaps in the spectrum of the Hill equation. Quart. Appl. Math. 11, 496–498 (1954).

Let $f(t)$ be real and of period 1 and have Fourier series $\sum c_n \exp(2\pi int)$. For Hill's equation $x''+(\lambda+f(t))x=0$ there exists a sequence of closed intervals, I_k: $\lambda_k\leq\lambda\leq\lambda^k$, $k=1, 2, \cdots$, where $\lambda_k<\lambda^k<\lambda_{k+1}$ such that Hill's equation has solutions bounded over $(-\infty, \infty)$ only if λ belongs to the closed set $S=\sum I_k$. Let $m(\lambda)$ for $-\infty<\lambda<\infty$ be defined by $m(\lambda)=\inf|\lambda-\mu|$ for μ in S. It is shown that $m^3(\lambda)\leq 2\sum_1^\infty|c_n|^2$ if $\lambda\geq -c_0$. From this follows

$$\lambda_{k+1}-\lambda^k\leq 2(2\sum_1^\infty|c_n|^2)^{\frac{1}{3}}$$

provided $\frac{1}{2}(\lambda_{k+1}+\lambda^k)\geq -c_0$.

N. Levinson.

Butlewski, Z. Sur les intégrales oscillantes d'une équation du second ordre. Bull. Soc. Amis Sci. Poznań. Sér. B. 11, 3–22 (1951). (Russian summary)

The author studies the oscillation and boundedness of solutions of a differential equation of the form

$$(1)\qquad (d/dt)[\theta(t,x)dx/dt]+f(t,x)=0.$$

Generalizations of known results for the corresponding linear case are obtained. A typical theorem is this. Let D be the domain $t\geq t_0>0$, $-\infty<x<\infty$, and suppose in D that $\theta>0$, θ, f, θ_t, θ_x are continuous, and that $\theta^{-1}(t,x)$, $f(t,x)$ satisfy Lipschitz conditions there, with respect to x. If $\theta_t\leq 0$ in D, $\theta_x\geq 0 (t\geq t_0>0, x>0)$, $\theta_x\leq 0 (t\geq t_0>0, x<0)$, $f(t,x)\uparrow$ with x, $xf(t,x)>0 (t\geq t_0>0, x\neq 0)$, $\int^x f(t,c)dt=c\cdot\infty$ $(c\neq 0)$, then the solution $x(t)$ of (1) defined by the conditions $x(t_1)\neq 0$, $\dot{x}(t_1)=0$ $(t_1\geq t_0)$ is oscillatory on the interval $t\geq t_1$.

W. Leighton (St. Louis, Mo.).

Watson, A. G. D. The Sturmian theory of oscillations. Math. Gaz. 38, 15–17 (1954)

Brock, Paul. The nature of solutions of a Rayleigh-type forced vibration equation with a large coefficient of damping. J. Appl. Phys. 24, 1004–1007 (1953).

The Rayleigh equation $\ddot{y}+\epsilon F(\dot{y})+y=A\cos\omega t$, with ϵ large, $A=a\epsilon$, $\omega=u/\epsilon$ (a and u of order 1) is considered for a piecewise linear F which is designed to approximate $\frac{1}{3}\dot{y}^3-\dot{y}$; such an approximation was studied, for $A=0$, by J. J. Stoker [Nonlinear vibrations . . . , Interscience, New York, 1950, pp. 141–147; these Rev. 11, 666]. The solution with $\dot{y}=1$, $y/\epsilon=\xi_0$ when $t=t_0$ is calculated explicitly, and two equations (p. 1007, (3.3a) and (3.3b)) for ξ_0, t_0 are found which will guarantee that this solution is periodic with period $2\pi/\omega$. The author then states, "If a forcing function is given, initial conditions for a periodic solution can be found."

G. E. H. Reuter (Manchester).

de Castro, Antonio. Sulle oscillazioni non-lineari dei sistemi in uno o più gradi di libertà. Rend. Sem. Mat. Univ. Padova 22, 294–304 (1953)

Setting up a curve in the $(x, \dot{x})$ plane which is cut by solutions of $\ddot{x}+f(x,\dot{x},t)+g(x)=0$ from outside to inside, the existence of a periodic solution follows in the familiar way when f is periodic in t. Extensions to a pair of equations are made. The function g is odd, $\neq 0$ for $x\neq 0$, increasing and $\to\infty$ as $x\to\infty$. The criteria on f are too long to present.

N. Levinson (Cambridge, Mass.).

Minorsky, Nicolas. Sur les systèmes non linéaires à deux degrés de liberté. C. R. Acad. Sci. Paris 238, 646–647 (1954).

This note relates to the system of equations

$$\ddot{x}_1+\omega_1^2x_1=(a-bx_1^2)\dot{x}_1+\epsilon Qx_2,$$
$$\ddot{x}_2+\omega_2^2x_2=(a-bx_2^2)\dot{x}_2+\epsilon Qx_1,$$

representing two coupled van der Pol oscillators. Assuming that a, b, ϵ, $|\omega_1-1|$, $|\omega_2-1|$ are small, the author employs the stroboscopic method to obtain an approximate description of the solutions.

L. A. MacColl (New York, N. Y.).

Fisher, Edward. The period and amplitude of the Van Der Pol limit cycle. J. Appl. Phys. 25, 273–274 (1954).

The amplitude and period of the limit cycle of the van der Pol equation, $\ddot{y}+y=\nu\dot{y}(1-y^2)$, are found for all values of ν by joining graphically the results of solutions about $\nu=0$ and $\nu=\infty$.

Author's summary.

Mizohata, Sigeru. Sur les phénomènes de sauts dans certains systèmes non linéaires. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **27**, 203–221 (1953).

A detailed analysis is made of the well-known "jump phenomenon" [cf. J. J. Stoker, Nonlinear vibrations . . . , Interscience, New York, 1950, pp. 94–96; these Rev. 11, 666] for the equation $\ddot{x}+\gamma P\dot{x}+x+\gamma Ax^3=\gamma E\cos\omega t$ (with $\omega^{-1}=1+\gamma T$, γ small). It is known that, for certain values of the parameters P, A, and T, the amplitude of a stable periodic solution may depend discontinuously on the forcing amplitude E. Roughly speaking, the author shows that such discontinuities really occur if E is a function of t which varies sufficiently slowly, and likewise if E is fixed but T is varied slowly. *G. E. H. Reuter* (Manchester).

Yamada, Hikoji. On the calculation of free oscillations with intermediate non-linearity. Reports Res. Inst. Appl. Mech. Kyushu Univ. **1**, 11–21 (1952).

The differential equation describing certain oscillations may be put in the form $\Delta(v,\sigma,\alpha)=vdv/d\sigma-f(\sigma,v,\alpha)=0$ where α is a linear function of the amplitude, v is the velocity, and σ is a non-dimensional displacement ranging between preassigned limits. Approximate solutions are found by assuming the form of $v(\sigma)$ and determining the coefficients so that $\Delta(v,\sigma,\alpha)$ vanishes for certain values of σ and $\int_a^b\Delta(v,\sigma,\alpha)\sigma^n d\sigma=0$ for $n=0,1,\cdots,N$. This method is applied to the equation of an oscillator with viscous damping and to the van der Pol equation. An iteration process is used to obtain a closer approximation to the periodic solution of the van der Pol equation for a particular value of the damping constant. *F. M. Stewart.*

***Flügge-Lotz, Irmgard. Discontinuous automatic control.** Princeton University Press, Princeton, N. J., 1953. viii+168 pp. $5.00.

In the simplest case a discontinuous automatic control system is a control system in which the correcting force is a positive constant A, or the corresponding negative constant $-A$, depending upon whether the sign of the error is positive or negative. Because of their simplicity, such systems are widely used, and the literature contains discussions of many particular systems of this kind. The present book represents the first attempt to treat such systems in a comprehensive and general way.

The book consists of four principal parts. The first of these is devoted to a study of the differential equation

$$\psi''+2D\psi'+\psi=\mp\operatorname{sgn}(\psi+K\psi'),$$

governing the motion of a body with one degree of freedom under discontinuous position control. The method used is the familiar one of phase-plane analysis, and the discussion is notable chiefly for its care and thoroughness. The second part is concerned with the system of equations

$$\psi''+2D\psi'+\psi=\beta,\quad \beta'=\mp\operatorname{sgn}(\psi+K\psi'+\mu\beta),$$

governing the motion of a body with one degree of freedom under discontinuous velocity control. Use is made of the phase space of the points (ψ,ψ',β), and much of the detail is concerned with devices for displaying configurations in the three-dimensional space by means of plane diagrams. The preceding differential equations relate to systems which are ideal, in the sense that certain imperfections which exist in actual systems have been neglected. The third part of the book contains a study of the effects of these imperfections. It turns out, paradoxically, that some of the imperfections may actually have favorable effects upon the motion. In the final part an attempt is made to extend the theory to cases of systems having more than one degree of freedom. However, this does not go much beyond the discussion of one rather restricted problem concerning the motion of an airplane or missile. The method used here depends on a combination of elementary analysis and the representation of a four-dimensional phase-space by means of two phase planes.

At first reading, the amount of intricate detail which the discussions involve is rather overwhelming, and a suspicion arises that large parts of the theory might have been presented somewhat more perspicuously. As yet the reviewer is undecided as to whether or not this suspicion is justified. In any event, the book constitutes a highly valuable contribution to the subject of automatic control, and it will, undoubtedly, lead to many further advances in the field. *L. A. MacColl* (New York, N. Y.).

***Draper, Charles Stark, McKay, Walter, and Lees, Sidney. Instrument engineering. Vol. II. Methods for associating mathematical solutions with common forms.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. xxviii+827 pp. $15.00.

This volume contains fourteen long chapters (Chapters 14 to 27), nearly two hundred illustrations, ten short tables, and one hundred twenty summaries varying from one to several pages in length. The volume weighs about five pounds. Chapter headings: Nondimensionalization of linear integro-differential equations; Operational methods of solving linear integro-differential equations with constant coefficients; The relating function, the weighting function, and the transfer function; Solutions associated with first-order differential equation forms; Solutions associated with second-order differential equation forms; Transient stability of mathematical solutions; Numerical methods of determining transient stability; Methods of solving third- and fourth-order characteristic equations; Graphical methods of determining transient stability; Graphical calculation of steady-state sinusoidal relating functions; Relating function forms from pulse function responses; Representation of solutions for differential equations with nonlinear terms of the rate-determined step type; Numerical analysis by the number series transformation method. There is a bibliography of over eighty references. The book deals with the nonhomogeneous ordinary differential equation of order n, with constant coefficients. The general discussion includes the integro-differential equation obtained by adding iterated integrals of the unknown function to the linear combinations of derivatives that appear in the equation. Methods of solving these equation are discussed and illustrated with considerable attention to special kinds of forcing functions including step and impulse functions. Stability of solutions or the comparison of the ouput with the forcing function is discussed at length, with much space devoted to the algebraic problem of determining properties of the roots of the characteristic equation. The determination of the forcing function corresponding to a prescribed output and treatments of some special nonlinear equations are also discussed. Formal methods are stressed; references are given to the theory upon which the methods are based. Notations, terminology, diagrams, and exposition seem to be more involved than necessary. Heading of Summary 20–3 (p. 385): "Absolute and relative stability in terms of characteristic equation root complex plane plots." *R. V. Churchill* (Ann Arbor, Mich.).

Doetsch, Gustav. Die lineare Differentialgleichung im zweiseitig unendlichen Intervall unter Anfangs- und Randbedingungen. Math. Ann. **126**, 307–324 (1953).

The differential equation considered here is the equation $\sum_{i=0}^{n} c_i Y^{(i)}(t) = F(t)$ on the infinite interval $-\infty < t < \infty$, where $F(t)$ denotes a continuous function and the coefficients c_i are constants with $c_n = 1$ such that the roots α_i $(i = 1, 2, \cdots, n)$ of the characteristic equation are distinct. If the two limits $F(\pm\infty)$ exist then there exists a solution of the differential equation with prescribed initial values $Y(-\infty)$, $Y'(-\infty)$, $\cdots$, $Y^{(n-1)}(-\infty)$ provided (a) that $\Re(\alpha_i) \neq 0$ $(i = 1, 2, \cdots, n)$ and (b) that

$$c_0 Y(-\infty) = F(-\infty), \quad Y'(-\infty) = \cdots = Y^{(n-1)}(-\infty) = 0.$$

Unless $F(t)$ satisfies further conditions, the conditions (a) and (b) are necessary for the existence of the solution. The solution is written in the form

$$Y(t) = \int_{-\infty}^{\infty} Q(t-\tau) F(\tau) d\tau + \sum_{(\mu)} k_\mu \exp(\alpha_\mu t),$$

where α_μ denotes those roots α_i for which $\Re(\alpha_i) > 0$, k_μ are arbitrary constants, and where the Green's function can be described simply in terms of the numbers α_i. Thus the solution is unique if and only if all roots α_i have negative real parts. When each k_μ is taken as zero the special solution $Y_0(t)$ satisfies the end conditions $c_0 Y_0(+\infty) = F(+\infty)$, $Y_0'(+\infty) = \cdots = Y_0^{(n-1)}(+\infty) = 0$. Also, the above formula with $k_\mu = 0$ represents the only solution of the differential equation that satisfies the boundary conditions $c_0 Y(-\infty) = F(-\infty)$, $c_0 Y(+\infty) = F(+\infty)$ under the hypotheses that $\Re(\alpha_i) \neq 0$ $(i = 1, 2, \cdots, n)$ and that the limits $F(\pm\infty)$ exist.

Similar results are given when the conditions on the limits of $F(t)$ are replaced by the condition of absolute integrability of that function over the infinite interval. The analysis leading to all these results involves properties of the two-sided Laplace transformation, properties which are established at the beginning of the paper.

R. V. Churchill (Ann Arbor, Mich.).

***Zachrisson, L. E. Problème aux limites posé à une seule extrémité dans quelques équations aux variations. The one-point boundary problem of some variation equations.** Actes du Colloque International des Vibrations non linéaires. Ile de Porquerolles, 1951, pp. 31–43, Publ. Sci. Tech. Ministère de l'Air, Paris, no. 281 (1953). (French and English)

This paper is concerned with various properties of the Cauchy function $u = u(t, \tau)$ of the linear homogeneous differential equation (*) $\sum_{\nu=0}^{n} a_\nu(t) d^\nu u/dt^\nu = 0$ determined by the one-point conditions $\partial^i u(t, \tau)/\partial t^i = \delta_{i,n-1}$ $(i = 0, 1, \cdots, n-1)$ at $t = \tau$. Special attention is given to the case in which (*) has a regular singular point at $t = 0$, and to the case in which (*) has a singular point of unit rank at $t = \infty$.

W. T. Reid (Evanston, Ill.).

Kaufman, H., and Sternberg, R. L. A two-point boundary problem for ordinary self-adjoint differential equations of even order. Duke Math. J. **20**, 527–531 (1953).

It is shown that the differential equation

$$(1) \qquad \sum_{0}^{n} D^j(p_{n-j}(x) D^j u) = 0$$

$(D = d/ds$, $n \geqq 2$, sufficiently differentiable real coefficients, $p_0 > 0)$ has no solution $\neq 0$ whose derivatives of order $< n$ vanish at two separate points with sufficiently large $x > 0$, provided that $x^{2-2n} p_0(x)$ is bounded away from zero for large $x > 0$ and that $\lim_{N\to\infty} \int^N x^{2i-1} p_i(x) dx$ exists and is finite when $i = 1, \cdots, n$. This extends a result by Hille on non-oscillatory behavior of solutions of Sturm-Liouville equations [Trans. Amer. Math. Soc. **64**, 234–252 (1948); these Rev. **10**, 376]. Interpreted properly, the result is also true when the coefficients of (1) are square matrices.

L. Gårding (Princeton, N. J.).

Naĭmark, M. A. Investigation of the spectrum and expansion in eigenfunctions of singular nonselfadjoint differential operators of the second order. Uspehi Matem. Nauk (N.S.) **8**, no. 4(56), 174–175 (1953). (Russian)

A number of results are stated concerning the spectrum of the operator L_θ associated with $l(y) = -y'' + p(x)y$ on $0 \leqq x < \infty$ with boundary condition $y'(0) - \theta y(0) = 0$ where the function p is complex-valued and θ is a complex constant. If $\int_0^\infty |p(x)| dx < \infty$, the spectrum of L_θ is continuous on the half-line $\lambda \geqq 0$ and otherwise discrete; the eigenvalues form a bounded set in the λ-plane with limit points on $\lambda \geqq 0$ only. If

(a) $\quad p(x) \to \infty$ as $x \to +\infty$;

(b) $\quad |p|' = O(|p|^\alpha)$, $0 < \alpha < 3/2$;

(c) $\quad |p'| = O(|p|')$, $|p''| = O(|p|'')$;

(d) $\quad 0 \leqq \arg p(x) \leqq \gamma$, $\gamma < \pi$,

then the spectrum is discrete and has no finite limit points. If (a), (b) and (c) hold and if $0 \leqq \arg p \leqq \pi$, if

$$\operatorname{Re}(p^{\frac{1}{2}}) = o(p^{-\frac{1}{2}}), \quad 0 \leqq \arg p^{\frac{1}{2}} \leqq \pi/2,$$

and if $\int_0^\infty |p|^{-\frac{1}{2}} dx = \infty$, then the continuous spectrum fills the whole real axis and for other values of λ the spectrum can only be discrete. Other results are stated.

N. Levinson (Cambridge, Mass.).

Nagumo, Mitio. Application of the calculus of variations to partial differential equations of the first order. J. Osaka Inst. Sci. Tech. Part I. **2**, 85–88 (1950). (Esperanto).

The solution of a partial differential equation of the form

$$(E) \qquad f(x_1, \cdots, x_n, u, \partial u/\partial x_1, \cdots, \partial u/\partial x_n) = 0$$

is related to the calculus of variations through the identification of the differential equations of the characteristic strips of (E) with the Euler equations of an associated variational problem of Lagrange type. *W. T. Reid* (Evanston, Ill.).

Hornich, Hans. Über lineare partielle Differentialgleichungen, deren Koeffizienten Polynome sind. Arch. Math. **4**, 437–440 (1953).

Soit l'équation aux dérivées partielles

$$\sum_{i_1, \cdots, i_n} P_{i_1, \cdots, i_n}(x_1, \cdots, x_n) \frac{\partial^{i_1 + \cdots + i_n} u}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} = f(x_1, \cdots, x_n),$$

où les $P_{i_1, \cdots, i_n}(x_1, \cdots, x_n)$ sont des polynômes à coefficients constants en $x_1, \cdots, x_n$ et le second membre, une série de Laurent $\sum f_{j_1 \cdots j_n} x_1^{j_1} \cdots x_n^{j_n}$, convergente pour $0 < |x_i| < \delta$, $\delta > 0$. L'auteur cherche à exprimer u par une série de Laurent convergente dans un voisinage de l'origine. La détermination de cette dernière série n'est possible que si on impose des conditions assez particulières aux coefficients des $P_{i_1, \cdots, i_n}$.

H. G. Garnir (Liège).

Duff, G. F. D. On linear partial differential equations of the second order having geodesic solutions. Canadian J. Math. **6**, 73–79 (1954).

A non-parabolic linear homogeneous partial differential equation of the second order can be written in the form

$$(1)\qquad \frac{1}{a^{1/2}}\sum_{i,k}\frac{\partial}{\partial x^i}\left(a^{1/2}a^{ik}\frac{\partial u}{\partial x^k}\right)+\sum_i b_i\frac{\partial u}{\partial x^i}+cu=0$$

where $a^{-1}=|\det a^{ik}|$, or $\Delta u+\mathbf{b}\cdot\operatorname{grad} u+cu=0$, where Δ is the Laplacian operator in the Riemannian space of metric (2) $\sum_{i,k}a_{ik}dx^idx^k$. Such a Riemannian space has been called completely harmonic [Copson and Ruse, Proc. Roy. Soc. Edinburgh **60**, 117–133 (1940); these Rev. **2**, 20] if the equation $\Delta u=0$ has a solution which depends only on the geodesic distance s from some fixed base point Q to the variable point P, no matter where Q may be.

It is shown here that the equation (1) has a solution which is a function of s alone, no matter where Q may be if and only if: (a) c is constant; (b) the vector $\mathbf{b}$ vanishes identically; (c) the space (2) is completely harmonic. And if these conditions hold, the equation has two such independent solutions, one regular near Q, the other the elementary solution with singularity at Q. *E. T. Copson.*

Ludford, G. S. S. Extensions in the applicability of Riemann's formula. J. Rational Mech. Anal. **3**, 77–88 (1954).

Cauchy's problem with irregular initial conditions is discussed for a hyperbolic differential equation

$$u_{xy}+au_x+bu_y+cu=0$$

for which a, b, c, a_y, b_y are continuous functions of x, y. The irregularities admitted are of the three following kinds: (1) a finite number of discontinuities in the initial values of u_x and u_y; (2) a finite number of discontinuous changes in direction of the initial curve C; (3) a finite number (>1) of points of intersection of some characteristics with C. A problem carrying irregularities of only the first two kinds is shown with the help of Riemann's function to have a unique, single-valued solution with piecewise continuous derivatives, provided the initial data satisfy certain integral conditions. A problem with the three kinds of irregularities need have no single-valued solution in the plane, but, if the initial data are subject to suitable integral conditions, does have a unique solution which is single-valued over a sheeted surface associated with C. *A. Douglis.*

Mihăileanu, N. N. Sur les invariants projectifs de l'équation de Laplace. Acad. Repub. Pop. Române. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **4**, 829–832 (1952). (Romanian. Russian and French summaries)

Si l'équation de Laplace $x_{uv}+ax_u+bx_v+cx=0$ possède des invariantes de la forme $h=(C_1uv+C_2u+C_3v+C_4)^{-2}$ elle conserve cette circonstance par une substitition homographique

$$u'=\frac{\alpha_1u+\alpha_2}{\alpha_3u+\alpha_4},\qquad v'=\frac{\beta_1v+\beta_2}{\beta_3v+\beta_4}.$$

Cette assertion est une conséquence directe de l'idée d'objet géométrique. L'expression h est invariante jusqu'aux constantes si les coefficients β_i sont donnés par

$$\begin{aligned}
\rho\beta_1&=C_2C_3\alpha_1-C_1C_3\alpha_2+C_3C_4\alpha_3-C_1C_4\alpha_4\\
\rho\beta_2&=C_2C_4\alpha_1-C_2^2\alpha_2+C_4^2\alpha_3-C_2C_4\alpha_4\\
\rho\beta_3&=C_1C_3\alpha_1+C_1^2\alpha_2-C_3^2\alpha_3+C_1C_3\alpha_4\\
\rho\beta_4&=C_1C_4\alpha_1+C_1C_2\alpha_2-C_3C_4\alpha_3+C_2C_3\alpha_4.
\end{aligned}$$

En ce cas $x_{uv}=p(C_1uv+C_2u+C_3v+C_4)^{-2}x$ est aussi invariante. L'équation généralisée de Poisson

$$x_{uv}-\frac{m-1}{C(Cu+D)}x_u-\frac{x}{(Cu+D)^2}=0$$

conserve les invariants h par le changement

$$u'=(u+\alpha)(-\alpha ur^2+1-2\alpha r)^{-1}.$$

Résumé de l'auteur.

Laskar, Williams. Généralisation de la méthode de factorisation de Schrödinger. C. R. Acad. Sci. Paris **238**, 772–774 (1954).

L'auteur a établi la méthode de factorisation [Infeld et Hull, Rev. Modern Physics **23**, 21–68 (1951); ces Rev. **13**, 239] sous une forme tout à fait générale par l'application des théorèmes sur les racines des opérateurs hermitiens dans l'espace de Hilbert. On peut obtenir les opérateurs de la méthode rationnellement en fonction de telles racines et on n'est plus limité à une seule variable comme Infeld et Hull.

T. E. Hull (Vancouver, B. C.).

Petiau, Gérard. Sur certaines solutions à singularités localisées des équations d'ondes des corpuscules en mouvement rectiligne et uniforme. C. R. Acad. Sci. Paris **238**, 998–1001 (1954).

Mikusiński, J. G.-. Sur un type de conditions mixtes pour les équations aux dérivées partielles. Studia Math. **13**, 277–286 (1953).

For the equation

$$(*)\qquad \sum_{\mu=0}^{m}\sum_{\nu=0}^{n}a_{\mu\nu}\frac{\partial^{\mu+\nu}}{\partial t^\mu\partial\lambda^\nu}x(t,\lambda)=\varphi(t,\lambda)$$

with $a_{\mu\nu}$ constant, the author considers the mixed initial-boundary-value problem. That is, the solution $x(t,\lambda_1)$ and $x(t,\lambda_2)$, $t\geqq 0$ is prescribed on the lines $\lambda=\lambda_1$ and $\lambda=\lambda_2$ while on the λ-axis m particular combinations of the unknown function and its first $m-1$ derivatives with respect to t are given. Equation (*) is transformed into the operational equation $a_0x^n+a_1x^{(n-1)}+\cdots+a_n=f(\lambda)$, where $a_{n-\nu}=\alpha_{m\nu}s^m+\cdots+\alpha_{0\nu}$. It is shown that a necessary and sufficient condition that the solution of (*) subject to the above conditions be unique is that the characteristic equation $a_0w^n+a_1w^{n-1}+\cdots+a_n=0$ have exactly two logarithmic roots w_1 and w_2 and that $w_2-w_1\neq 2k\pi i/(\lambda_2-\lambda_1)$, $k=\pm1,\pm2,\cdots$. The author uses techniques developed previously for ordinary equations [Studia Math. **12**, 227–270 (1951); these Rev. **13**, 751]. Several examples illustrating the theorem are given. *M. H. Protter.*

Pini, Bruno. Sul problema di Dirichlet per le equazioni lineari del secondo ordine di tipo ellittico nei domini non limitati. Rend. Accad. Sci. Fis. Mat. Napoli (4) **19** (1952), 157–170 (1953).

Let $x=x(s)$, $y=y(s)$, $0\leqq s\leqq L$ be the parametric representation of a simple closed curve C with continuous curvature. The infinite domain D bounded by C contains the family of curves $C(t)\equiv(x=x(s)+ty'(s),\ y=y(s)-tx'(s))$ for t sufficiently small and positive. Dirichlet-type problems associated with D and the elliptic equation

$$(*)\qquad L(u)\equiv u_{xx}+u_{yy}+a(x,y)u_x+b(x,y)u_y+c(x,y)u=f(x,y)$$

usually assign point conditions on the boundary C and at infinity. In the present paper the assigned conditions are in

general of an integral nature both on C and at infinity, and are associated with a positive function $H(x, y)$ of class C^2. The coefficients of the differential equation (*) are assumed to be bounded functions with a, $b \in C^1$ and c, $f \in C^0$. A function $u(x, y) \in C^2$ in D is said to be a solution of the generalized Dirichlet problem if $L(u)=f$ interior to D and

$$\lim_{t\to 0} \int_0^{2\pi} H(u-\varphi)^2 c(t) ds = 0,$$

where the given function $\varphi(s) \in \mathfrak{L}^2$ on $0 \leqq s \leqq L$. The author shows that if there is a positive constant μ such that $|H_x| < \mu H$, $|H_y| < \mu H$, $M(H)+cH \leqq 0$, then in the class of functions $u \in C^2$ in D such that $\lim_{t\to 0} \int_0^{2\pi} (Hu^2)_{\gamma(t)} d\theta = 0$ there exists at most one solution of the generalized Dirichlet problem. Here $M(u)$ is the operator adjoint to $L(u)$ and $\gamma(t)$ is the circle with center at $(0, 0)$ and radius $1/t$. The existence of a solution of the generalized Dirichlet problem is proved under a few additional conditions, the outstanding one being that for some $\epsilon (0 < \epsilon < 1)$ $\lim_{t\to 0} \int_0^{2\pi} (H^\epsilon)_{\gamma(t)} d\theta = 0$. Similar results are obtained for the infinite strip region $-\infty < x < \infty$, $\delta_1(x) < y < \delta_2(x)$. *F. G. Dressel.*

Pini, Bruno. Un problema di valori al contorno per l'equazione

$$\frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 u}{\partial y^2} = 0.$$

I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 609–615, 746–749 (1953).

Let the two functions $\chi_i(y)$ $(i=1, 2; \chi_1(y) < \chi_2(y))$ have continuous derivatives on $a \leqq y \leqq b$; and denote with the letter D the region $a \leqq y \leqq b$, $\chi_1(y) \leqq x \leqq \chi_2(y)$. The function $\omega = \omega(x, y)$ is assumed to have on D the properties: $\omega[\omega_x]$ is continuous as a function of x, y and absolutely continuous with respect to $y[x]$, and $(\omega_{xx})^2$, $(\omega_y)^2$ are summable. The following boundary-value problem is shown to have a solution. There exists a function $V = V(x, y)$ which interior to D satisfies the equation $\partial^4 V/\partial x^4 - \partial^2 V/\partial y^2 = 0$, on the boundary of D the function $V = \omega$, and if (x, y) is an interior point of D the following limit holds almost everywhere $\lim_{(x,y)\to(\chi_i(y),y)} V_x(x, y) = \omega_x(\chi_i(y), y)$. If ω satisfies certain additional conditions the words "almost everywhere" can be deleted from the preceding statement. At the end of the paper it is stated that similar results can be proved for solutions $V = V(x_1, \cdots, x_n, y)$ of the equation

$$\left(\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{\partial}{\partial y}\right)\left(\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - \frac{\partial}{\partial y}\right) V = 0.$$

F. G. Dressel (Durham, N. C.).

Pini, Bruno. Sui punti singolari delle soluzioni delle equazioni paraboliche lineari. Ann. Univ. Ferrara. Sez. VII. (N.S.) **2**, 1–12 (1953).

Assume $x = \chi_i(y)$, $i=1, 2$, are two functions with finite derivatives on $a \leqq y \leqq b$, and such that $\chi_1(y) < \chi_2(y)$ for $a < y \leqq b$. The letter D will be used to denote the region $a \leqq y \leqq b$, $\chi_1(y) \leqq x \leqq \chi_2(y)$; and S will denote the boundary of D exclusive of the line segment $\chi_1(b) < x < \chi_2(b)$. Let $U(x, y) = y^{-1/2} \exp(-x^2/4y)$, $y > 0$, and set

$$\omega(x, y; \alpha, \mu) = [1 + \sum_{k=1}^{n} U^\alpha(x-x_k, y-y_k)] \exp(\mu y),$$

where each point (x_k, y_k) is on S. The function $c(x, y)$ in the differential equation $L(u) \equiv u_{xx} - u_y + c(x, y)u = 0$ is assumed to be continuous and the letter m used below denotes a number such that $\max |c(x, y)| < m$. The author shows that a solution $u = u(x, y)$ in D of $L(u) = 0$ vanishes if the ratio u/ω behaves in certain ways in the neighborhood of the boundary S of D. In more detail the results are (i) if point (x, y) is interior to D and if $\lim_{(x,y)\to(\bar{x},\bar{y})} u(x, y)/\omega(x, y; l; m) = 0$ for every point $(\bar{x}, \bar{y})$ of S, then $u \equiv 0$ in D; (ii) set $\gamma_i(y, t) = \chi_i(y) - (-1)^i t$, $\omega = \omega(x, y; \alpha, m)$, if $p > 1$, $0 < \alpha < (p-1)/p$ and if

$$\lim_{t\to 0} \int_{\gamma_1(y,t)}^{\gamma_2(y,t)} \left|\frac{u}{\omega}\right|^p dx = \lim_{t\to 0} \sum_{i=1}^{2} \int_{a'}^{b'} \left|\frac{u}{\omega}\right|_{\gamma_i(y,t)}^p dy = 0,$$

the limit in the first integral holding uniformly for sufficiently small positive t, and the limit in the second integral holding for all a', b' such that $a < a' < b' < b$, then necessarily $u \equiv 0$ in D. Results similar to (i) and (ii) are obtained for several types of unbounded regions in the (x, y)-plane, and for solutions of the equation

$$\sum_{i=1}^{n} u_{x_i x_i} - u_y + c(x_1, \cdots, x_n, y)u = 0.$$

F. G. Dressel (Durham, N. C.).

Katz, S., and Peiser, A. M. A system of non-linear partial differential equations arising in heat transfer. J. Math. Physics **32**, 256–268 (1954).

The authors consider the system of partial differential equations

$$f(u)\frac{\partial u}{\partial x} = u - v = g(v)\frac{\partial v}{\partial y} \tag{1}$$

with the boundary conditions $u(0^+, y) = \varphi(y)$, $v(x, 0^+) = \psi(x)$, $x > 0$, $y > 0$, which describe the transfer of heat from a fluid to a bed of particles, through which the fluid is flowing; x is the distance along the bed, y is the time, u and v represent the temperatures of the fluid and of the bed. For the case that f and g are constants the second of the present authors has given the solution in a former paper [Proc. Amer. Math. Soc. **1**, 650–661 (1950); these Rev. **13**, 554]. Here the general case is considered. Throughout the paper f and g are considered as continuous functions satisfying the inequalities:

$$|f(t)| \geqq \alpha > 0, \quad |g(t)| \geqq \alpha > 0, \quad -\infty < t < +\infty.$$

The differential equations are replaced by the difference equations

$$f(U_{m,n})(U_{m+1,n} - U_{m,n})|_h = U_{m,n} - V_{m,n} = g(V_{m,n})(V_{m,n+1} - V_{m,n})|_k \tag{2}$$

with the conditions $U_{0,n} = \varphi(nk)$, $V_{m,0} = \psi(mh)$, $m, n = 0, 1, 2, \cdots$. In the first place the existence of a unique solution of the system (1) is proved by means of considerations about the system of integral equations

$$\int_{\varphi(y)}^{u(x,y)} f(s)ds = \int_0^x \{u(\xi, y) - v(\xi, y)\}d\xi,$$

$$\int_{\psi(x)}^{v(x,y)} g(s)ds = \int_0^y \{u(x, \eta) - v(x, \eta)\}d\eta,$$

where $x \geqq 0$, $y \geqq 0$. In the second place it is proved that the solution of the system (2) converges to the desired solution, if we put $m = [x/h]$, $n = [y/k]$ and $h \to 0$, $k \to 0$. Good error estimates for the approximation given by the solution of the system (2) seem difficult. With the additional assumption of a Lipschitz-condition the authors find estimates which grow exponentially with x and y. In the last section,

however, they give for the important case that f and g are constants of opposite sign a much smaller estimate.

H. Bremekamp (Delft).

Lauwerier, H. A. Optimum problems in the conduction of heat in a semi-infinite solid. Appl. Sci. Research A. **4**, 142–152 (1953).

The initial temperature of a solid filling the region $x \geqq 0$ is zero. The face $x=0$ is maintained at a positive constant temperature T_0 during the time interval $0<t<t_0$ and kept insulated after the instant $t=t_0$. Corresponding values of t_0 and T_0 are sought such that the temperature at a prescribed depth x will reach a prescribed value T_m $(T_m<T_0)$ with a minimum input of heat energy per unit area at the face $x=0$. The corresponding investigation is made when the condition of constant surface temperature during the period $0<t<t_0$ is replaced by the condition that the flux of heat into the solid at the surface is a constant Q_0 during that period. The author indicates that the minimum energy required to raise the temperature at depth x to T_m, for all possible t_0 and T_0, is attained when the surface temperature or flux is an impulse (heat explosion). Methods of calculus are applied to the well-known temperature formulas to derive graphical results. [Reviewer's comment: the author's description and justification of his process of getting optimum values are inadequate.] *R. V. Churchill* (Ann Arbor, Mich.).

Lauwerier, H. A. Diffusion from a source in a skew velocity field. Appl. Sci. Research A. **4**, 153–156 (1953).

The author deals with a problem of determining steady-state concentrations $c(x, y)$ of a substance in a fluid flowing with velocity $v=v_0(1+ay)$ in the direction of the x axis when a source of the diffusing substance is present at the origin and when $c=0$ at $y=-1/a$. He assumes that the equation of diffusion is $Dc_{yy}=vc_x$. [He states no conditions under which the equation assumes that form, nor does he explain the particular boundary conditions he introduces to represent the source at the origin. He uses the two-sided Laplace transformation formally but follows it by an inverse transformation based on one-sided transforms.]

R. V. Churchill (Ann Arbor, Mich.).

Integral Equations

Fantappiè, Luigi. Calcolo esatto degli autovalori e delle autofunzioni di un nucleo "variato," per una variazione di tipo elementare. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 458–462 (1951).

The kernel of a Fredholm integral equation

$$y(s)=\lambda\int_a^b K(s,t)y(t)dt+f(s)$$

is perturbed such that $K^*=K(s,t)+\epsilon H(s,t)$, with $\epsilon=$const. and with $H(s,t)=\sum_{i=1}^{n}p_i(s)q_i(t)$, is the new kernel. The author proposes to calculate the eigenvalues of K^* as the poles of $\Gamma(s,t,\lambda,\epsilon)$ (reciprocal kernel to K^*) and the eigenfunctions from the well known residuum integral formulas for $\Gamma(s,t,\lambda,\epsilon)$. If $\Gamma(s,t,\lambda,0)$ is known, then

$$\Gamma(s,t,\lambda,\epsilon)-\Gamma(s,t,\lambda,0)$$

can be calculated as a degenerate kernel by means of simple formulas. *H. Bückner* (Schenectady, N. Y.).

Wall, H. S. Concerning continuous continued fractions and certain systems of Stieltjes integral equations. Rend. Circ. Mat. Palermo (2) **2**, 73–84 (1953).

For functions $f(x)$, $g(x)$, continuous and of bounded variation on $[a, b]$, it is shown that there is on the square $a\leqq x, t\leqq b$ a unique solution (A, B, C, D) of the system of Stieltjes integral equations

$$A(x,t)=1+\int_a^t C(s,t)df(s),\quad B(x,t)=\int_a^t D(s,t)df(s),$$

$$C(x,t)=\int_a^t A(s,t)dg(s),\qquad D(x,t)=1+\int_a^t B(s,t)dg(s),$$

and representations of the functions A, B, C, D in terms of uniformly and absolutely convergent series are obtained. For the continuous continued fraction $\Phi_a^b(f, g, z)$ recently introduced by Puig Adam [Revista Mat. Hisp.-Amer. (4) **11**, 180–190 (1951); these Rev. **13**, 540] and subsequently by A. Stöhr [Math. Nachr. **6**, 103–107 (1951); **8**, 157–165 (1952); these Rev. **14**, 24, 855], the author shows in this case that $\Phi_a^b(f, g, z)$ exists and satisfies

$$\Phi_a^b(f,g,z)=\frac{A(a,b)z+B(a,b)}{C(a,b)z+D(a,b)},$$

for all z for which $C(a, b)z+D(a, b)\neq 0$. Also, for arbitrary z and for t in $[a, b]$ there is a subinterval of $[a, b]$ containing t in which the unique solution $y(x)$ of the Riccati-Stieltjes integral equation

$$y(x)=z-\int_t^x df(s)+\int_t^x y^2(s)dg(s)$$

is $y=\Phi_z^t(f, g, z)$. *W. T. Scott* (Evanston, Ill.).

Bicadze, A. V. A spatial analogue of the Cauchy-type integral and some of its applications. Doklady Akad. Nauk SSSR (N.S.) **93**, 389–392 (1953); errata, **94**, 980 (1954). (Russian)

The substance of this note is contained in the paper reviewed below. *W. J. Trjitzinsky* (Urbana, Ill.).

Bicadze, A. V. Spatial analogue of an integral of Cauchy type and some of its applications. Izvestiya Akad. Nauk SSSR. Ser. Mat. **17**, 525–538 (1953). (Russian)

Let D be a three-dimensional domain, bounded by a closed Liapounoff surface S, and $q=q_1i+q_2j+q_3k$ be an assigned vector, whose components q_s are functions of (x, y, z) and have first-order partials continuous in the closure $\bar{D}$ of D. Let $\rho=\rho(\xi,\eta,\zeta;x,y,z)$ be the distance between (ξ, η, ζ), (x, y, z) and introduce vectors p_1, p_2, p_3, of which

$$p_1=\rho^{-3}\{[(\xi-x)q_1-(\eta-y)q_2-(\zeta-z)q_3]i+[(\xi-x)q_2+(\eta-y)q_1]j+[(\xi-x)q_3+(\zeta-z)q_1]k\}$$

is typical. Green's formula is applied to the p_s. Fundamental is the following three-dimensional analogue of a formula of D. Pompeiu [Rend. Circ. Mat. Palermo **33**, 108–113 (1912); **35**, 277–281 (1913)] for functions of a complex variable: $\iint_S Mqd\omega+\iiint_D(\text{div } q\cdot\text{grad } \rho^{-1}+\text{rot } q\times\text{grad } \rho^{-1})d\tau=4\pi q$ (in D), $=0$ (in D'), where D' is the complementary domain and M is a certain matrix, involving the normal to S. A vector q is potential in D if div $q=$rot $q=0$ in D; for such a vector one has the Cauchy formula $(4\pi)^{-1}\iint_S Mqd\omega=q$ (in D), $=0$ (in D'). On the basis of the above fundamental formulas the author studies effectively the corresponding integrals of Cauchy type, under H (Hölder) conditions, the boundary values of such integrals (analogues of Plemelj formulas), interchange of principal integrals (analogue of the Poincaré-

Bertrand formula); these considerations enable inversion of the singular integral equation $(2\pi)^{-1}\int\int_S M\phi d\omega=\psi$, where ψ ($\in H$) is assigned on S. Further, the author considers the Poisson formula for a sphere, the Dirichlet problems for the half-space $z>0$ and for a space, cut along a circular disc, as well as inversion of an integral equation relating to contact problems of three-dimensional elasticity.

W. J. Trjitzinsky (Urbana, Ill.).

Bicadze, A. V. Inversion of a system of singular integral equations. Doklady Akad. Nauk SSSR (N.S.) **93**, 595–597 (1953); errata, **94**, 980 (1954). (Russian).

Under the hypotheses and with the notations employed in the two papers reviewed above, a study is made of the system

$$(1)\quad A\phi(P_0)+B(2\pi)^{-1}\int\int_S M(P_0, Q)\phi(Q)d\omega_Q=f(P_0) \qquad (P_0, Q \text{ on } S),$$

where A, B are constant matrices (of 4^2 elements), $f=(f_1, \cdots, f_4)$ is a vector of class H on S; vector ϕ is to be found in H on S. Integration in (1) is in the sense of principal values. It is shown that, if $\det(A+B)\neq 0$, $\det(A-B)\neq 0$ and if $G=(A+B)^{-1}(A-B)$ is a matrix whose consecutive rows are (g_1, g_2, g_3, g_4), $(-g_2, g_1, -g_4, g_3)$, $(-g_3, g_4, g_1, -g_2)$, $(-g_4, -g_3, g_2, g_1)$, the system (1) has a unique solution which can be given explicitly with the aid of a Cauchy-type three-dimensional integral. *W. J. Trjitzinsky.*

Mendes, Marcel. Systèmes d'équations intégrales et figures dérivées des ellipsoïdes hétérogènes en rotation. J. Math. Pures Appl. (9) **32**, 335–386 (1953).

The author had studied previously [same J. (9) **24**, 51–72 (1945); these Rev. **8**, 291] figures infinitely near ellipsoids of rotation when the latter consist of a finite number of homogeneous layers of densities increasing from the surface to the interior. In that work application was made of the method of Poincaré and of Lamé's functions, in the case when the ellipsoids of separation are homofocal; this method does not apply conveniently when the ellipsoids of separation are homothetic. In the present work the author studies the problem by means of the ordinary formulas of the potential, first for homofocal ellipsoids and then for homothetic ellipsoids. By resolving a system of linear homogeneous integral equations it is established, in each case, that there exists a figure infinitely near the one from which one starts. The author finds it necessary to make a substantial study of a certain system of linear integral equations, to which it is possible to extend, in an appropriate manner, the notion of symmetric kernels and the Schmidt theory; such a study has interest in itself, even apart from the use made of it in resolving the problem under consideration. Use is made of kernels "symmetric" in the sense that $K_{ij}(x, y)=K_{ji}(y, x)$. Of importance is a group of functions $f_i(x)=\sum_{\nu=1}^{r}\int_a^b K_{i\nu}(x, s)h_\nu(s)ds$, where the kernels are "symmetric". These functions are developed according to the characteristic functions; with the aid of such developments the author proves the vanishing of certain constants which occur in the integral equations resolving the problem. This enables the solution of the problem of a single ellipsoid and the rediscovery of the results of Poincaré. Further, there is an effective and complete treatment of the cases when the ellipsoids of separation are homofocal and when they are homothetic. *W. J. Trjitzinsky* (Urbana, Ill.).

Gianuizzi, Maria. Problemi di valori al contorno per equazioni integro-differenziali: teoremi di esistenza e di unicità. Ann. Univ. Ferrara. Sez. VII. (N.S.) **2**, 71–91 (1953).

Let $Y(x)=[y(x), y'(x), \cdots, y^{(n-1)}(x)]$ for $a\leqq x\leqq b$. The author proves the existence of a unique solution of the integro-differential equation

$$y^{(n)}(x)=f\left[x, Y(x), \int_a^x g(x, u, Y(u))du\right],$$

subject to the initial conditions $y(x_i)=c_i$, where

$$a\leqq x_1<x_2<\cdots<x_n\leqq b, \quad \text{and} \quad c_1, \cdots, c_n$$

are real numbers. The central hypotheses are that for fixed x and u, the functions f and g satisfy Lipschitz conditions with small coefficients. Thus the author can verify the hypotheses of the theorem of Hildebrandt and Graves [Trans. Amer. Math. Soc. **29**, 127–153 (1927)] embodying the process of successive approximations, and a special case of this theorem sometimes called the Caccioppoli fixed-point theorem. In the second part of the paper, a similar equation with two independent variables is treated. *R. G. Bartle.*

Okabe, Jun-ichi. An illustrative example of solving a sum equation by Fourier transforms. Rep. Res. Inst. Appl. Mech. Kyushu Univ. **2**, 155–160 (1953).

The equation in question is an infinite difference equation in two variables which the author reduces to an equation in one variable by means of the Fourier transform. This equation is equivalent to an infinite set of linear algebraic equations in an infinite number of unknowns. These equations are solved formally, ignoring the rather involved questions of convergence. *E. Pinney* (Berkeley, Calif.).

Functional Analysis, Ergodic Theory

Orlicz, W. On functions of finite variation, depending on a parameter. Studia Math. **13**, 218–232 (1953).

Let X be a real Banach space, $x(v)$ a function defined from a real interval $[a, b]$ to X. If $A\subset[a, b]$ the variation and strong variation of x over A are respectively defined as

$$V_A(x)=\sup_\pi\left\|\sum_{i=1}^{n-1}\epsilon_i[x(v_{i+1})-x(v_i)]\right\|,$$

$$W_A(x)=\sup_\pi\sum_{i=1}^{n-1}\|x(v_{i+1})-x(v_i)\|,$$

where the supremum is taken over all partitions π with points in A and for $\epsilon_i=\pm 1$. If $X=M$, the space of essentially bounded functions over $[a, b]$, it is shown that

$$V^*(x)=\sup_{|A|=0}V_A(x)=\operatorname{ess\,sup}_t[\operatorname{ess\,var}_v D(t, v)],$$

where $x(t)=D(:, v)$. Various necessary and sufficient conditions for functions to be of bounded variation are derived. In particular, if $\int_a^b D(t, v)dt=\int_a^b D(t, u)dt$, bounded variation of $D(:, v)$ is equivalent to a generalized Lipschitz condition. As an application of the general theorems derived, conditions are given on real sequences $\{\lambda_i\}$ which insure that if $\sum_{i=1}^{\infty}a_i\varphi_i(t)$ is a Fourier expansion of a function in M, then $\sum_{i=1}^{\infty}\lambda_i a_i\varphi_i(t)$ shall also be a Fourier expansion of a function in M. *R. E. Fullerton* (Madison, Wis.).

Klee, V. L. Jr. On a theorem of Béla Sz.-Nagy. Amer. Math. Monthly 60, 618–619 (1953).

Let B and C be two bounded closed convex bodies in a normed linear space L, each including the origin 0 in its interior. Let f be the "radial" mapping of B on C, which is defined in the following way: $f(0)=0$; if $0\neq x\in B$, then $f(x)=[b(x)/c(x)]x$, where $b(x)$ and $c(x)$ denote the gauge functionals (Distanzfunktion) for B and C, respectively, i.e., for example, $b(x)=\inf\{t: t>0, t^{-1}x\in B\}$. Using elementary properties of the gauge functionals, a simple proof is given of the following theorem: f is Lipschitzian, i.e., $\|f(x)-f(y)\|\leq M\|x-y\|$ $(x, y\in B)$ with a finite constant M. This theorem was (for euclidean L) first observed by the reviewer; proofs were given by Vincze and Szűsz [Acta Sci. Math. Szeged 14, 96–100 (1951); these Rev. 12, 573] and by Haupt [S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1951, 147–161 (1952); these Rev. 14, 305]. *B. Sz-Nagy.*

Kadec, M. I. On homeomorphism of certain Banach spaces. Doklady Akad. Nauk SSSR (N.S.) 92, 465–468 (1953). (Russian)

In a Banach space B a sequence of linearly independent elements $\{x_n\}$ is called a T-system if for each y in B and each integer n the value $|E_n(y)|=\inf_{t_k}\|y-\sum_{0\leq k<n}t_kx_k\|$ is assumed for a unique choice of the real numbers $t_k=t_k^{(n)}$. Then choose the sign of $E_n(y)$ to be the sign of $t_n^{(n+1)}$ if this is not zero, to be the sign of $E_{n+1}(y)$ if $t_n^{(n+1)}=0$. It is asserted that given c_j, $j=1, \cdots$, a sufficient for the existence of a y such that $c_j=E_j(y)$, all $j\geq 0$, is that the sequence satisfies the three conditions (a) $\lim_j c_j=0$, (b) $|c_{j-1}|\geq|c_j|$ for all j, and (c) $|c_{j-1}|=|c_j|$ implies $c_{j-1}=c_j$. If each such sequence determines a unique y, the space B is said to satisfy Bernstein's condition. A page is devoted to the proof that a space isomorphic to c_0 satisfies this condition; it is noted that the usual basis in l_p, $p\geq 1$, shows that all these spaces satisfy this condition. These facts are applied to prove c_0 homeomorphic to l_1 by means of the theorem: If two Banach spaces satisfy Bernstein's condition, then they are homeomorphic. *M. M. Day* (Urbana, Ill.).

Wermer, John. On restrictions of operators. Proc. Amer. Math. Soc. 4, 860–865 (1953).

The author considers a bounded linear invertible operator τ on a complex Banach space B, and a closed subspace C whose image under τ is properly contained in C and which has a single generator x (that is $\{\tau^n x\}$ is fundamental in C). Let T denote the restriction of τ to C. The author constructs a homomorphism of C onto a certain space J of functions analytic in D (the component of the resolvent set of τ containing the origin). The kernel of the homomorphism is C^∞, the intersection of the ranges of T^n $(n=1, 2, \cdots)$, and the operator T is represented by multiplication by the complex variable z. If C^∞ reduces to the zero element, the author shows further that any bounded linear operator U on C which commutes with T is represented on J by multiplication by a function $u(z)$ of J. If U is not the zero operator, its null-space consists of the zero element alone. In particular, no non-trivial projection can commute with T.

In case B is a Hilbert space, the author denotes by C^0 the orthogonal complement of C^∞ in C. If τ is a normal operator whose spectrum lacks interior and separates the plane into exactly two components, he shows that both C^0 and C^∞ are invariant under T, that the restriction of T to C^∞ is normal, and that the restriction of T to C^0 can be isomorphically represented as multiplication by z on a certain Hilbert space J of analytic functions. If, further, τ is unitary, J can be taken to be H^2, the space of functions $F(z)$ analytic in $|z|<1$ with $\int_{-\pi}^{\pi}|F(re^{i\theta})|^2d\theta$ bounded on $r<1$. *A. F. Ruston* (Sheffield).

Ritt, R. K. A condition that $\lim_{n\to\infty}n^{-1}T^n=0$. Proc. Amer. Math. Soc. 4, 898–899 (1953).

The author proves the following theorem: Let T be a bounded linear transformation on a Banach space; let the spectrum of T be interior to the unit circle, with the possible exception of $z=1$; further, suppose that there is an $M>0$ and an $\eta>0$ such that if z is in the resolvent set of T, $|z|\geq 1$, and $|z-1|\leq\eta$, then $\|(z-1)(z-T)^{-1}\|\leq M$. Then $\lim_{n\to\infty}n^{-1}T^n=0$. The proof uses the method of contour integration from the Cauchy theory of analytic functions of T and is based on a characteristically delicate choice of contours. *E. R. Lorch* (Rome).

Leżański, T. The Fredholm theory of linear equations in Banach spaces. Studia Math. 13, 244–276 (1953).

Fredholm determinants for linear functional equations in normed linear complete spaces have been set up by A. D. Michal and R. S. Martin [J. Math. Pures Appl. (9) 13, 69–91 (1934)] and more recently by A. F. Ruston [Proc. Lond. Math. Soc. (3) 1, 327–384 (1951); 3, 368–377 (1953); these Rev. 13, 468; 15, 134]. In this paper, for the functional equation $\varphi-\lambda K\varphi=\psi$, it is assumed that the elements φ and ψ are in a linear closed subspace Ξ of X^* the conjugate space of the linear normed space X, K is a member of a class $\mathfrak{K}$ of linear transformations on Ξ to Ξ, which contains the identity I, as well as one-dimensional transformations of the form $M\varphi=[K\varphi, x_0]\varphi_0$, for all K of $\mathfrak{K}$, φ_0 of Ξ and x_0 of X. There is also postulated a linear continuous functional F on $\mathfrak{K}$ to real numbers and a linear transformation T on Ξ to Ξ, such that $F(M)=F([K\varphi, x_0]\varphi_0)=[KT\varphi_0, x_0]$. If the symbol $K\begin{pmatrix}\varphi_1\cdots\varphi_n\\ x_1\cdots x_n\end{pmatrix}$ represents the determinant of $[K\varphi_i, x_j]$, $i, j=1, \cdots, n$, then it is shown that the expression

$$F_{\varphi_1x_1}F_{\varphi_2x_2}\cdots F_{\varphi_nx_n}K\begin{pmatrix}\varphi_1\varphi_2\cdots\varphi_n\\ x_1x_2\cdots x_n\end{pmatrix}$$

has significance as a real number and the expression

$$F_{\varphi_1x_1}F_{\varphi_2x_2}\cdots F_{\varphi_nx_n}K\begin{pmatrix}\varphi\varphi_1\cdots\varphi_n\\ xx_1\cdots x_n\end{pmatrix}$$

is an element of $\mathfrak{K}$. Moreover, by using the Hadamard determinant theorem it is possible to show that

$$\left|F_{\varphi_1x_1}F_{\varphi_2x_2}\cdots F_{\varphi_nx_n}K\begin{pmatrix}\varphi_1\cdots\varphi_n\\ x_1\cdots x_n\end{pmatrix}\right|\leq\|F\|^n\|K\|^n\eta^{\frac{1}{2}n}.$$

As a consequence

$$D(\lambda K)=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}F_{\varphi_1x_1}\cdots F_{\varphi_nx_n}K\begin{pmatrix}\varphi_1\cdots\varphi_n\\ x_1\cdots x_n\end{pmatrix}$$

converges for all λ as does also

$$D_n\left(\lambda K;\begin{matrix}\psi_1\cdots\psi_m\\ y_1\cdots y_m\end{matrix}\right)=(-1)^m\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}F_{\varphi_1x_1}\cdots F_{\varphi_nx_n}K\begin{pmatrix}\psi_1\cdots\psi_m\varphi_1\cdots\varphi_n\\ y_1\cdots y_mx_1\cdots x_n\end{pmatrix}.$$

The usual Fredholm theory is then possible for a functional equation of the form $\varphi-\lambda KT\varphi=\psi$. Among the special cases considered there is that in which X is L^p and $\Xi=X^*$ is $L^{p'}$, with $p+p'=pp'$, $1<p<\infty$. The class $\mathfrak{K}_0$ consists of

transformations of the form $\int K(s,t)y(t)dt$ where $K(s,t)$ is such that $\int\int x(s)K(s,t)y(t)dsdt$ exists for all x of L^p and y of $L^{p'}$. The class $\mathfrak{K}$ is $\mathfrak{K}_0$ extended to include the identity transformation I. T is determined by a

$$T(s,t)=\int R(s,r)S(r,t)dr$$

with $\int(\int|R(s,r)|^p ds)^{u/p}dr<\infty$ and

$$\int\left(\int|S(r,t)|^{p'}dt\right)^{v/p'}dr<\infty$$

with $u+v=uv$, and $F(K)=\int\int K(s,t)T(t,s)dsdt$. In particular, the Fredholm determinant of $T(s,t)$ exists with $\int T(s,s)ds$ suitably defined. This is reminiscent of T. Carleman [Math. Z. 9, 196–217 (1921)] for the case $p=u=2$. The case where $X=l^p$ and $X^*=l^{p'}$ is similar [see L. W. Cohen, Bull. Amer. Math. Soc. 36, 563–572 (1930)]. Other instances considered are: X the space of almost bounded functions, $\mathfrak{T}$ the space L^1; X the space of functions of bounded variation, $\mathfrak{T}$ the space of continuous functions. For the case where X is the space m of bounded sequences, $\mathfrak{T}$ the space l^1 of absolutely convergent sequences, it is assumed that the class $\mathfrak{K}$ consists of infinite matrices such that $\sum_{ij}x_i k_{ij}y_j$ converges for all x_i in m and y_j in l^1, while T is an infinite matrix t_{ij} such that $\sup_j\sum_i|t_{ij}|<\infty$, with $F(K)=\sum_{ij}k_{ij}t_{ij}$. In this case the Fredholm determinant of T defined in the paper reduces to the absolutely convergent infinite determinant of $\delta_{ij}+t_{ij}$. [See L. W. Cohen, Ann. of Math. (2) 37, 326–335 (1936).] It is not shown in what way the hypothesis on F induces properties of the transformation T, though in the illustrations considered T is always chosen to be completely continuous so that KT is also. There is also a seeming contradiction with the convergence properties of the Michal-Martin determinant to which the one in this paper can be reduced. *T. H. Hildebrandt.*

Vaĭnberg, M. M. On solvability of certain operator equations. Doklady Akad. Nauk SSSR (N.S.) 92, 457–460 (1953). (Russian)

The author continues his earlier work on non-linear equations [same Doklady (N.S.) 73, 253–255 (1950); 92, 213–216 (1953); Uspehi Matem. Nauk (N.S.) 7, no. 4(50), 55–102 (1952); these Rev. 12, 111; 15, 439; 14, 384]. The principal results are the following. Theorem 1. In a real Hilbert space H let B be a positive, self-conjugate linear operator and let $F(x)=\text{grad}\,f(x)$, where f satisfies the condition $2f(x)\leqq a_1(x,x)+a_2(x,x)^\gamma+a_3$, where a_2 and a_3 are positive numbers, $0<\gamma<1$, and $0<a_1\|B\|<1$. Suppose further that (a) f is continuous and B is compact or (b) f is weakly upper semi-continuous. Then the equation $x=BF(x)$ has a solution. Theorem 2. Suppose: (1) B is self-conjugate in H, the positive part of the spectrum of B is in an interval (m,n), $m>0$, and the range of E_n-E_m is finite-dimensional, where $\{E_t\}$ is the spectral resolution of B; (2) $F(x)=\text{grad}\,f(x)$, where $f(x)\geqq(x,x)/m+a_2(x,x)^\gamma+a_3$, where a_2 and a_3 are positive numbers and $0<\gamma<1$; and (3) (a) above or (b′) f is weakly lower semicontinuous. Then $x=BF(x)$ has a solution. An application to non-linear integral equations is discussed in which are needed results of the first references above. *M. M. Day* (Urbana, Ill.).

Richter, Vladimir. Gewöhnliche Differentialgleichungen für Funktionen mit Werten in einem Banach-Raum. Arch. Math. 4, 477–484 (1953).

An existence theorem of the Peano type is given for the differential system $y'(\xi)=f(\xi,y(\xi))$, $y(\xi_0)=y_0$, where ξ varies over a real interval and y varies over a compact set in a Banach space P, the values of $f(\xi,y)$ also lying in P. The hypotheses are such that the proof can be carried over from the ordinary case. As an application there is an existence theorem for the infinite system

$$\frac{d\eta_i}{d\xi}=\varphi_i(\xi,\eta_1,\eta_2,\cdots),\quad \eta_i(\xi_0)=a_i,\quad i=1,2,\cdots,$$

where $\{\eta_i\}$ and $\{\alpha_i\}$ are points of the space l^p $(p>1)$. Here the compact set in l^p is defined by $|\eta_i-\alpha_i|\leqq\beta_i$ with $\{\beta_i\}$ a point of l^p. The functions φ_i are assumed continuous and to satisfy a condition: $|\varphi_i(\xi,\eta_1,\eta_2,\cdots)|<m\beta_i$, $m>0$. Generalizations of other standard theorems, including a uniqueness theorem of Kamke, are also given. *A. E. Taylor.*

Owchar, Margaret, and Tingley, Arnold J. On the absolute convergence of a Fourier-Hermite expansion of nonlinear functionals. Proc. Amer. Math. Soc. 5, 85–88 (1954).

The authors give sufficient conditions for the convergence of the Fourier-Hermite development of a functional $F(x)$ belonging to $L_2(c)$ and having the further properties that it is continuous in the Hilbert topology on C and satisfies a condition of the form $|F(x)|\leqq A\exp\{B\int_0^1[x(t)]^2dt\}$ for $A\geqq 0$, $B\geqq 0$. Their condition involves variations of suitably high order k of the functional F with respect to a subsequence of k functions of the sequence

$$y_i(t)=2^{\frac{1}{2}}\int_0^t\cos[(2i-1)\pi s/2]ds.$$

The proof makes use of a result of Cameron and Hatfield [same Proc. 4, 375–387 (1953); these Rev. 14, 989]. In comparing their present sufficient condition with sufficient conditions recently found by R. E. Graves [ibid. 4, 95–101 (1953); these Rev. 14, 636] the authors point out that in some respects his conditions are weaker, and in other respects stronger, than their present one. *W. T. Martin.*

Cameron, R. H., and Fagen, R. E. Nonlinear transformations of Volterra type in Wiener space. Trans. Amer. Math. Soc. 75, 552–575 (1953).

In an earlier paper [same Trans. 66, 253–283 (1949); these Rev. 11, 116], R. H. Cameron and the reviewer studied the behaviour of Wiener integrals under transformations of the form $T: y(t)=x(t)+\Lambda(x|t)$ where Λ is a non-linear functional of smooth variation in a Hilbert neighborhood of each point of a set Γ. In the present paper the authors point out that functionals Λ which are of the "Volterra type", i.e. which depend on integrals with variable upper limit, do not usually satisfy this condition of smooth variation and, in order to be able to handle functionals of this type, they first define functionals of semi-smooth variation. Having this definition they extend the transformation theorems obtained earlier to transformations involving functionals of semi-smooth variation. Their result is also more general in the sense that they require only that their conditions on Λ and on the kernel $K(x|s,t)$ of the variation of Λ shall hold in a uniform neighborhood of each point of the region under consideration instead of a Hilbert neighborhood. The proofs of their results are carried out in successive stages. First they apply the results of the above mentioned paper to a family of suitably smoothed transformations. Then by a limiting argument they obtain their formulas locally under slightly more restrictive conditions on K. These restrictions are then removed to obtain their desired results. *W. T. Martin* (Cambridge, Mass.).

***Platone, Giulio. Teoria della composizione col peso $A(y)$ sommabile nell'insieme delle funzioni pseudolimitate. Composizione ordinaria nell'insieme $[l_A]$ ed $[_Al]$.** Società Italiana per il Progresso delle Scienze, XLII riunione, Roma, 1949, Relazioni, Vol. primo, pp. 203–211. Società Italiana per il Progresso delle Scienze, Roma, 1951.

E is a Lebesgue measurable set in the Euclidean space S^r of dimension r. $A(y)$ is a function L-integrable on E. The class $[l]$ consists of functions $k(x, y)$ on $E\times E$ bounded except for a set of zero measure. Then the class $[l]$ forms a ring under pointwise addition with multiplication or composition $(k_1*k_2)_A$ defined as $\int_E A(y)k_1(x, t)k_2(t, y)dt$. Power series $\sum_{n=1}^{\infty}a_n z^n$ have as analogues $\sum_{n=1}^{\infty}a_n\lambda^n k^{(n)}$ where $k^{(n)}=(k*k^{(n-1)})_A$. The Volterra-Perez theory of permutable functions is then extensible to $[l]$. In particular, if the functional equations $f_p(z_1, z_2, \cdots, z_m)=0$, $p=1\cdots m$, with f_p expansible in power series around $(0, \cdots, 0)$ are solvable for $z_1, \cdots, z_m$ in terms of $z_{m-1}, \cdots, z_n$, then the expressions in $k_1, \cdots, k_n$ analogous to the power series are similarly solvable for $k_1, \cdots, k_m$. The spaces $[l_A]$ and $[_Al]$ consist of functions of the form $k(x, y)A(y)$ and $A(x)k(x, y)$, respectively, with $k(x, y)$ in $[l]$ and $A(y)$ L-integrable, and also form rings in which multiplication is ordinary composition.

T. H. Hildebrandt (Ann Arbor, Mich.).

Fantappiè, Luigi. Le variazioni e i funzionali derivati degli autovalori e delle autofunzioni di un nucleo dato. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 10, 375–379 (1951).

Ce travail donne les démonstrations de quelques-uns des résultats présentés par l'auteur dans une communication antérieure [Atti 3° Congresso Un. Mat. Ital., Pisa, 1948, Perrella, Roma, 1951, pp. 111–114].

Soit ϕ l'espace des fonctions $f(z)$ localement analytiques à l'origine et T une application linéaire continue de cet espace dans lui-même. D'après la théorie des fonctionnelles analytiques, l'équation $(I-\lambda T)\varphi=f$, où $\varphi(z)$ est l'inconnue, peut s'écrire

$$(\alpha)\qquad \varphi(z)-\frac{\lambda}{2\pi i}\int_C k\left(z, \frac{1}{\tau}\right)\frac{1}{\tau}\varphi(\tau)d\tau=f(z),$$

$k(z, t)$ étant la fonction "indicatice symétrique" de T et C étant, par exemple, une circonférence de centre 0 contenue dans un domaine d'holomorphie de $\varphi(z)$. Sous des restrictions précisées dans un travail précédent [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 9, 603–608 (1929)], l'auteur montre que la théorie des équations (α) est très semblable à celle des équations de Fredholm. Pour $\lambda\neq\infty$ le spectre de T se réduit à l'ensemble des valeurs propres, qui sont les zéros d'une certaine fonction entière $D(\lambda)$, "fonctionnelle déterminante de $k(z, t)$." Pour chaque zéro simple λ_r de $D(\lambda)$ on a une solution propre φ_r de T et une solution propre ψ_r de la transposée T' de T, déterminées à moins de facteurs constants. Or, quand le noyau $k(z, t)$ varie, λ_r, φ_r et ψ_r deviennent des fonctionnelles (analytiques) de k; et les variations $\delta\lambda_r$, $\delta\varphi_r$ et $\delta\psi_r$, correspondantes a une variation donnée δk du noyau, sont à leur tour des fonctionnelles linéaires de δk. Il s'agit maintenant de calculer ces fonctionnelles linéaires ou, ce qui revient au même, leurs indicatrices symétriques—"les fonctionnelles dérivées (symétriques) de λ_r, φ_r, ψ_r par rapport à la fonction k". On a d'abord la formule

$$\mathring{\mathfrak{D}}^n\lambda_r=\frac{1}{2\pi i}\int_{C_r}\lambda\mathring{\mathfrak{D}}^{n-1}\Gamma(\lambda; z, t)d\lambda,$$

où $\mathring{\mathfrak{D}}$ est l'opérateur de dérivation par rapport à k, $\Gamma(\lambda; z, t)$ l'indicatrice de la transformation $(I-\lambda T)^{-1}$ et C_r une courbe entourant chaque valeur de λ_r; les expressions des dérivées successives du "noyau associé" Γ de k ont été déduites dans une note précédente [ibid. (8) 5, 329–333 (1948); ces Rev. 11, 117]. Enfin l'auteur déduit les expressions de $\mathring{\mathfrak{D}}_k\varphi_r$ et $\mathring{\mathfrak{D}}_k\psi_r$.

D'après l'auteur ces recherches pourraient présenter de l'intérêt en physique théorique, en ce qui concerne la détermination des niveaux énergetiques d'atomes d'hydrogène.

J. Sebastião e Silva (Lisbonne).

Henriksen, Melvin. On the prime ideals of the ring of entire functions. Pacific J. Math. 3, 711–720 (1953).

Let R be the ring of entire functions. If $f\in R$, and $\{a_n\}$ is the sequence of distinct zeros of f, arranged so that $|a_1|\leq|a_2|\leq\cdots$, define $O_n(f)$ to be the multiplicity of a_n as a zero of f. If f has zeros but is not identically zero, define $m(f)=\sup_{n\geq 1}O_n(f)$. If $f(z)\equiv 0$, define $m(f)=\infty$. It is shown that a prime ideal P of R is non-maximal if and only if $m(f)=\infty$ for each $f\in P$. The existence of non-maximal prime ideals was first noted by Kaplansky. An ideal P is called fixed if there is some point z_0 such that $f(z_0)=0$ for each $f\in P$. Otherwise P is called free. The prime, fixed ideals are maximal. Each non-maximal prime ideal is contained in a unique maximal ideal (which is free). If M is a maximal free ideal, let P^* be the intersection of $M, M^2, \cdots$. Then P^* is prime, and it is the largest non-maximal prime ideal contained in M.

Let P_M be the class of prime ideals contained in M. By considering rates of growth, there is established in M an equivalence relation $\sim$ and an order relation $\ll$ such that if $f, g\in M$, one and only one of the relations $f\sim g$, $f\ll g$, $g\ll f$ holds. If Ω is any subset of M, the set of $f\in M$ such that $g\ll f$ for all $g\in\Omega$ is shown to be a prime ideal. Further, any element P of P_M arises in this way by taking $\Omega=M-P$. As a consequence, it is proved that the elements of P_M are linearly ordered under set inclusion.

The last part of the paper is concerned with residue class rings of prime ideals. If P is a prime ideal, R/P is a valuation ring whose unique maximal ideal is principal. If P is non-maximal, R/P is Noetherian if and only if $P=P^*$ (where P^* arises from the maximal M corresponding to P). Finally, R/P^* is isomorphic with the ring of formal power series over the complex field.

A. E. Taylor.

Haplanov, M. G. On the spectral theory of matrices in an analytic space. Doklady Akad. Nauk SSSR (N.S.) 90, 969–972 (1953). (Russian)

The principal theorem states that if $f_n(x)$ is a sequence of functions analytic in $|x|<R$ with a matrix mapping the analytic space A_{R_1} into A_R, $R_1<R$ [cf. Haplanov, same Doklady (N.S.) 80, 21–24, 177–180 (1951); these Rev. 13, 470, 357], then the sequence $h_n(x)=x^n-\lambda f_n(x)$ is a quasi-power basis in $|x|<R$ for any λ which is not an eigenvalue of the integral equation

$$\varphi(x)=f(x)+\lambda\int_C K(x, s)\varphi(s)ds,$$

where $K(x, s)=(2\pi i)^{-1}\sum_{j=1}^{\infty}\overline{f_j(x)}/s^{j+1}$, and C is any circle $|x|=r<R$. Similar problems relating to fundamental systems and normal bases in analytic spaces are reduced to problems about the spectra of integral equations.

B. Crabtree (Durham, N. H.).

Rožanskaya, N. N. On the point character of the spectrum of a certain class of matrices in an analytic space. Doklady Akad. Nauk SSSR (N.S.) **92**, 7–10 (1953). (Russian)

From the result of Haplanov stated in the preceding review it follows that if the matrix M of the sequence $f_n(x)$ maps A_R into A_R, $R_1<R$, then M has a spectrum consisting of isolated eigenvalues, i.e. a pure point spectrum. The present paper generalizes this result as follows: If the infinite matrix (a_{ik}) satisfies the condition $|a_{ik}|\leqq Nq^k$ for $i<k$, $0<q<1$, N an arbitrary real number, then the spectrum of (a_{ij}) consists entirely of isolated eigenvalues.

B. Crabtree (Durham, N. H.).

Kac, M., Murdock, W. L., and Szegö, G. On the eigenvalues of certain Hermitian forms. J. Rational Mech. Anal. **2**, 767–800 (1953).

Suppose that $f(x,\theta)$ is real-valued, periodic in θ with period 2π, and defined for $0\leqq x\leqq 1$, that the coefficients $\psi_n(x)$ of the Fourier expansion

$$f(x,\theta)\sim\sum_{n=-\infty}^{\infty}\psi_n(x)e^{in\theta}$$

are continuous, and that

$$\sum_{n=-\infty}^{\infty}\max_{0\leqq x\leqq 1}|\psi_n(x)|\leqq M.$$

Form the generalized Toeplitz matrix

$$A_n=\left[\psi_{\nu-\mu}\left(\frac{\mu+\nu}{2n+2}\right)\right]\quad(0\leqq\mu\leqq n,\ 0\leqq\nu\leqq n),$$

and let its eigenvalues, arranged in increasing order, be $\lambda_1^{(n)},\lambda_2^{(n)},\cdots,\lambda_{n+1}^{(n)}$. The authors show that if $F(\lambda)$ is any Riemann integrable function defined in $-M\leqq\lambda\leqq M$, then

$$\lim_{n\to\infty}\frac{F(\lambda_1^{(n)})+F(\lambda_2^{(n)})+\cdots+F(\lambda_{n+1}^{(n)})}{n+1}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^1 F[f(x,\theta)]\,dx\,d\theta.$$

This means that, roughly speaking, the $\lambda_\nu^{(n)}$ behave asymptotically like the values assumed by $f(x,\theta)$ at $n+1$ points (x,θ) uniformly distributed in the rectangle $-\pi\leqq\theta\leqq\pi$, $0\leqq x\leqq 1$.

In the case when $f(x,\theta)=f(\theta)$ is independent of x, the analogue of the above result is already known [G. Szegö, Math. Természz. Ért. **35**, 185–222 (1917)]; the following refinement of it is now proved. Let $f(\theta)$ be continuous and periodic with period 2π, let $\min_\theta f(\theta)=f(\theta_0)=m$, and let θ_0 be the only value of θ (mod 2π) at which this minimum is attained; suppose also that $f(\theta)$ has a continuous second derivative in a neighborhood of θ_0, and let $f''(\theta_0)\neq 0$. For ν fixed and $n\to\infty$ we then have

$$\lambda_\nu^{(n)}-m\sim f\left(\theta_0+\frac{\nu\pi}{n}\right)-f(\theta_0)\sim\frac{\nu^2\pi^2c}{n^2},$$

where $c=\frac{1}{2}f''(\theta_0)$.

The authors then prove the following theorem, which includes a continuous analogue of Szegö's result. Let $\rho(x)$ be an even function in $L(-\infty,\infty)$, and let its Fourier transform $F(\xi)=\int_{-\infty}^{\infty}\rho(x)e^{i\xi x}dx$ also belong to $L(-\infty,\infty)$. Let $K(x)$ be an even function, continuous almost everywhere, belonging to $L^2(-\infty,\infty)$, and such that $|K(x)|<M$ for all x. Let $\{\lambda_n(\alpha)\}$ be the sequence of eigenvalues of the integral equation

$$\int_{-\infty}^{\infty}K(\alpha x)\rho(x-y)K(\alpha y)\varphi(y)dy=\lambda\varphi(x),$$

and let $N_\alpha(a,b)$ denote the number of eigenvalues lying in (a,b). If (a,b) does not contain 0 and the sets of points

$$\{(x,\xi):K^2(x)F(\xi)=a\},\quad\{(x,\xi):K^2(x)F(\xi)=b\}$$

are both of measure 0, then

$$\lim_{\alpha\to 0}\alpha N_\alpha(a,b)=(2\pi)^{-1}\Omega(a,b),$$

where $\Omega(a,b)$ is the measure of the set

$$\{(x,\xi):a<K^2(x)F(\xi)<b\}.$$

As an application of this, it is shown that if $r_n(n)$ is the nth positive zero of the Bessel function $J_n(x)$, then

$$\lim_{n\to\infty}n^{-1}r_n(n)=1+x_1^3,$$

where x_1 is the first positive root of the equation $\tan x=x$.

Some variants and special cases of these results are discussed, and applications to probability theory are indicated. *F. Smithies* (Cambridge, England).

Brown, Arlen. On a class of operators. Proc. Amer. Math. Soc. **4**, 723–728 (1953).

Those operators (=bounded linear transformations) A of Hilbert space H are studied, for which A and A^*A commute. Equivalent definitions: a) In the polar decomposition $A=WP$, W and P commute; b) $A=VP=PV$ with $P\geqq 0$ and V isometric; c) $A=VB=BV$ with B self-adjoint and V isometric. If such an A is completely continuous, or if it generates a ring of operators which is finite in the sense of Murray-Neumann, then A is necessarily normal. In the general case, A need not be normal, and its structure may be characterized as follows. Let Z be the null-space of A. There exists a unique decomposition $H=H_1\oplus H_2$, reducing A, $A=A_1\oplus A_2$, such that (i) $Z\subset H_2$, (ii) A_2 is normal, (iii) A_1 is unitarily equivalent to the "deleted shift operator" $\hat{P}_0$ defined by a positive operator P_0, i.e., to the operator

$$\hat{P}_0\{x_0,x_1,x_2,\cdots\}=\{0,P_0x_0,P_0x_1,\cdots\}$$

of a Hilbert space $L\oplus L\oplus L\oplus\cdots$, P_0 being a positive operator in L. *B. Sz.-Nagy* (Szeged).

Halmos, Paul R. Commutators of operators. II. Amer. J. Math. **76**, 191–198 (1954).

[For part I see same J. **74**, 237–240 (1952); these Rev. **13**, 563.] If A and B are operators, i.e. bounded linear transformations, on complex Hilbert space H, then $A'=AB-BA$ is the commutator of A and B. Kaplansky has conjectured that if $A'A=AA'$, then $\|(A')^n\|^{1/n}\to 0$ as $n\to\infty$. Corroboration but no proof is adduced by showing that if $A'A=AA'$, then A' is a generalized divisor of zero. [See Schäffer, Anais Acad. Brasil. Ci. **25**, 87–90 (1953); these Rev. **15**, 39.] In the next section it is shown that if H is infinite-dimensional then the commutators are strongly dense in all operators. It is not known whether they are uniformly dense. The $\inf\{\|I-C\|\mid C\text{ commutator}\}$ has not yet been determined, although if H is finite-dimensional then $\|I-A'\|\geqq 1$ for each operator A and if H is infinite-dimensional $\|I-A'\|^2\leqq .97$ for some operator A. For H infinite-dimensional it is not known whether there is an operator A such that $I-A'$ is nilpotent. Every operator A on infinite-dimensional H can

be factored $A=PQ$ such that QP is a commutator. Every operator on an infinite-dimensional H can be written as the sum of two commutators. *S. Sherman.*

Dixmier, J. Formes linéaires sur un anneau d'opérateurs. Bull. Soc. Math. France **81**, 9–39 (1953).

It is well-known that certain results for rings of operators (W^*-algebras) bear an analogy to results in the theory of integration and can be regarded as extensions of the latter. This paper is primarily concerned with such extensions and accordingly overlaps slightly with a recent paper of I. E. Segal [Ann. of Math. (2) **57**, 401–457 (1953); these Rev. **14**, 991]. However the methods and approach are quite different.

In §1 some general results are obtained which are of independent interest. Let M, N be two W^*-algebras, $\mathfrak{m}$ a 2-sided ideal in M and θ: $A\to\theta(A)$ a linear mapping of $\mathfrak{m}$ into N. If $A\geqq 0$ implies $\theta(A)\geqq 0$, then θ is said to be "positive." If θ is positive, then θ is said to be "normal" if for any increasing filter $\mathfrak{F}$ of self-adjoint elements of $\mathfrak{m}$, with $A\in\mathfrak{m}$ as a least upper bound, the operator $\theta(A)$ is the least upper bound of $\theta(\mathfrak{F})$. In particular these concepts are applied to linear functionals defined on M. Now denote by M_* the class of all linear functionals f on M of the form $f(A)=\sum_{i=1}^{\infty}(Ax_i, y_i)$, where $\{x_i\}$, $\{y_i\}$ are fixed sequences of vectors in the Hilbert space such that $\sum\|x_i\|^2<\infty$, $\sum\|y_i\|^2<\infty$. Then M_* is a closed subspace of the B-space dual M^* of M and moreover M is the dual of M_* relative to the bilinear form induced on $M\times M_*$ by the canonical bilinear form in $M\times M^*$. Every element of M_* is a linear combination of positive functionals in M_* and a positive functional on M is normal if, and only if, it belongs to M_*. The following are some corollaries to the above. (1) The notion of a sub W^*-algebra of a W^*-algebra is purely algebraic. (2) If θ is a normal *-homomorphism of one W^*-algebra M into a second N, then $\theta(M)$ is a W^*-algebra. (3) If φ is a normal positive functional on M and θ_φ is the canonical representation of M associated with φ, then $\theta_\varphi(M)$ is a W^*-algebra. A number of results involving the various topologies for operator algebras are also obtained.

The non-commutative integration is done in §§2, 3. Let $\mathfrak{m}$ be a 2-sided ideal in M and denote by $\mathfrak{m}^+$ the non-negative self-adjoint operators in $\mathfrak{m}$. For $0<\alpha<\infty$, there exists an ideal $\mathfrak{m}^\alpha$ such that $(\mathfrak{m}^\alpha)^+$ consists of all A^α where $A\in\mathfrak{m}^+$ [Dixmier, Compositio Math. **10**, 1–55 (1952); these Rev. **14**, 481]. Denote by $\mathfrak{m}^0$ the strong closure of $\mathfrak{m}$ and by $\mathfrak{m}^\infty$ the ideal generated by the projections in $\mathfrak{m}$. Then $0\leqq\alpha\leqq\beta\leqq\infty$ implies $\mathfrak{m}^\beta\subset\mathfrak{m}^\alpha$. Now let φ be a normal trace defined on $\mathfrak{m}$ which is "faithful" in the sense that $\varphi(A)=0$ for $A\in\mathfrak{m}^+$ implies $A=0$. Then $\varphi(|A|)$, where $|A|=(A^*A)^{1/2}$, defines a norm on $\mathfrak{m}$. Denote the completion of $\mathfrak{m}$ with respect to this norm by L^1. If L^∞ is the completion of $\mathfrak{m}^0$ with respect to the norm $\|A\|$, then L^∞ is the dual of L^1. Next let $A\in\mathfrak{m}^{1/p}$ $(p\geqq 1)$ and define $\|A\|_p=[\varphi(|A|^p)]^{1/p}$. Then the Hölder inequality holds in the following form: Let $p, q, \cdots, r$ be numbers in $[1, \infty]$ such that

$$p^{-1}+q^{-1}+\cdots+r^{-1}=1.$$

If $A\in\mathfrak{m}^{1/p}$, $B\in\mathfrak{m}^{1/q}$, $\cdots$, $C\in\mathfrak{m}^{1/r}$, then

$$|\varphi(AB\cdots C)|\leqq\varphi(|AB\cdots C|)\leqq\|A\|_p\|B\|_q\cdots\|C\|_r.$$

Here φ need not be faithful. The Minkowski inequality follows to give that $\|A\|_p$ is a semi-norm on $\mathfrak{m}^{1/p}$, $1\leqq p\leqq\infty$. If φ is faithful, then $\|A\|_p$ $(1\leqq p)$ is a norm in $\mathfrak{m}^{1/p}$ and the completion of $\mathfrak{m}^{1/p}$ with respect to this norm is denoted by L^p. Then for $1<p<\infty$ and $p^{-1}+q^{-1}=1$, L^p is the dual of L^q. If A, B are non-negative self-adjoint operators in $\mathfrak{m}^{1/p}$, $\mathfrak{m}^{1/q}$ respectively where $1<p<\infty$ and $p^{-1}+q^{-1}=1$, then $\varphi(AB)=\|A\|_p\|B\|_q$ implies that A^p and B^q are proportional. For $p\geqq 2$, L^p is uniformly convex.

Finally let N be a sub-W^*-algebra of the W^*-algebra M. In §4 is proved the existence of a certain mapping of M into N which, in case M is of finite type and N is the center of M, reduces to the canonical mapping "$\natural$" obtained elsewhere by the author [loc. cit. and Ann. Sci. Ecole Norm. Sup. (3) **66**, 209–261 (1949); these Rev. **11**, 370]. *C. E. Rickart.*

Dixmier, J. Sous-anneaux abéliens maximaux dans les facteurs de type fini. Ann. of Math. (2) **59**, 279–286 (1954).

It has been well known that in certain rings of operators in Hilbert space there exist maximal abelian subrings which are not conjugate in any sense; in particular, this has been known for the approximately finite factor of Murray and von Neumann. The present author studies this situation in more detail. Let M be a factor of type II_1 and M_1 a maximal abelian (self-adjoint) subring of M. Let U be the group of those unitary operators $V\in M$ for which $VM_1V^*=M_1$ and P the (weakly closed) ring generated by U. Then it is clear that $M_1\subseteq P\subseteq M$. The author proves that if M is approximately finite then there exist subrings M_1 for which $P=M_1$, others for which $P=M$ and also subrings M_1 for which P is a factor not equal to M. The proof consists of explicit elementary constructions with M as the weakly closed group algebra of a certain discrete group and M_1 chosen as the subring generated by a suitable subgroup.

F. I. Mautner (Baltimore, Md.).

Griffin, Ernest L., Jr. Some contributions to the theory of rings of operators. Trans. Amer. Math. Soc. **75**, 471–504 (1953).

Généralisation de certains résultats de Murray et von Neumann aux anneaux d'opérateurs semi-finis (i.e., sans partie de type III). Soit M un anneau d'opérateurs. L'auteur définit l'invariant C de M et M': 1) lorsque M et M' sont finis, explicitant ainsi les résultats de Kaplansky [C. R. Acad. Sci. Paris **231**, 485–486 (1950); ces Rev. **12**, 186] (C est considéré comme un opérateur du centre de M); 2) lorsque M et M' sont semi-finis (C est alors un "opérateur formel" qui résume l'existence de certains cardinaux attachés aux "parties infinies" de M et M'). L'auteur établit à cette occasion quelques théorèmes de structure sur les anneaux semi-finis. Il est prouvé que C joue un rôle important dans les questions suivantes, étudiées en détail dans la suite du mémoire: conditions pour qu'un isomorphisme soit spatial, représentation d'une forme linéaire positive complètement additive, comparaison des topologies forte et ultraforte, faible et ultrafaible, conditions pour qu'un anneau d'opérateurs soit standard, continuité des isomorphismes. Comme l'auteur le signale, certains résultats ont été obtenus indépendamment par R. Pallu de la Barrière [ibid. **234**, 795–797 (1952); ces Rev. **13**, 756]. L'article de Dye [Trans. Amer. Math. Soc. **72**, 243–280 (1952); ces Rev. **13**, 662], qui est indiqué dans la bibliographie, pourrait être cité parfois de façon plus explicite. Le terme "semi-fini" est bien choisi, mais le terme "purement infini" risque de prêter à confusion. *J. Dixmier* (Paris).

Yood, Bertram. Topological properties of homomorphisms between Banach algebras. Amer. J. Math. **76**, 155–167 (1954).

Let B be a real Banach algebra. An involution on B is a mapping $x\to x^*$ for which $(x+\lambda y)^*=x^*+\lambda y^*$ for λ real,

$x^{**}=x$, and either $(xy)^*=x^*y^*$ or $(xy)^*=y^*x^*$. Call B a ρ^*-algebra if B has an involution and a non-negative subadditive function $|\ |$ which vanishes only at 0 and satisfies $|x|<\rho(x)$ for x self-adjoint or skew ($\rho(x)$ is the spectral radius of x in the complexification of B). If moreover $|\ |$ can be chosen so that B is an (F)-space (in the sense of Banach) under the metric $|x-y|$, then B is a ρ_e^*-algebra. The A^*-algebras of Rickart [Ann. of Math. (2) **51**, 615–628 (1950); these Rev. **11**, 670] are ρ^*-algebras. The involution in a ρ^*-algebra is continuous. ρ^*-algebras are semi-simple and have unique norm. A homomorphism of a Banach algebra onto a self-adjoint subalgebra of a ρ^*-algebra is continuous. If B is a ρ_e^*-algebra for which each self-adjoint element has a real spectrum and T is an isomorphism of B into a Banach algebra, then T preserves the spectral radius of self-adjoint and skew elements; if moreover there is a $c>0$ such that $\|T(x^*)\|\leq c\|T(x)\|$ for all x, then T^{-1} is continuous on $T(B)$.

Now let B (B_1) be a complex commutative Banach algebra with radical R (R_1) and let $\mathfrak{M}'$ ($\mathfrak{N}'$) be its set of maximal regular ideals (including B (B_1) itself). Let T be a homomorphism of B into B_1. For $N\in\mathfrak{N}'$ let $T^*(N)$ be the unique $M\in\mathfrak{M}'$ determined by the rule $x[T^*(N)]=T(x)(N)$ for $x\in B$. If B is regular, in the sense of Šilov [Trav. Inst. Math. Stekloff **21** (1947); these Rev. **9**, 596], then $T^*(\mathfrak{N}')$ consists of the maximal regular ideals containing $T^{-1}(R_1)$ ($T^{-1}[T(R)]$); for $R=0$ and T one-to-one the second result reduces to one due to Rickart [Proc. Amer. Math. Soc. **4**, 191–196 (1953), thm. 1; these Rev. **14**, 660]. If B is regular and is closed under uniform limits when considered as an algebra of functions on the set of maximal regular ideals containing $T^{-1}(0)$ and if $T^{-1}(0)$ is the intersection of the maximal regular ideals containing it, then the closure of $T(B)$ is $T(B)\oplus R_1$ and the canonical isomorphism of $B/T^{-1}(0)$ onto $T(B)$ has a continuous inverse.

The paper contains other results and most of the above results are proved in a more general setting.

J. A. Schatz (Bethlehem, Pa.).

Have

+Maeda, Fumitomo. Renzoku kikagaku [Continuous geometry]. Iwanami Shoten, Tokyo, 1952. 2+3+225 pp. 600 Yen.

This book contains nearly all the important facts which are known today about J. von Neumann's continuous geometries [Continuous geometry, I, II–III, Inst. Advanced Study, Princeton, 1936, 1937]. Proofs are complete and may readily be followed by a beginning student. The first three chapters (1. Fundamental concepts about lattices; 2. General properties of modular lattices; 3. Projective spaces)—nearly one third of the volume—are devoted to an introduction to the algebraic theory of lattices with some special considerations on complete lattices in which the operations are continuous. The continuity of operations is formulated in terms of directed sets; the equivalence of this formulation with J. von Neumann's and with another more algebraic one is proved in the appendix to the book, in which also the equivalence between the axiom of choice, the well-ordering theorem and Zorn's lemma are shown by one of the well-known methods. The cumbersome transfinite induction, which was used repeatedly by J. von Neumann, is, in this book, replaced by Zorn's lemma.

The next two short chapters exhibit the theory of dimension and sub-direct decomposition of a continuous geometry for the case where no group of automorphisms is considered —only I. Halperin's [Ann. of Math. (2) **40**, 581–599 (1939); these Rev. **1**, 45] and F. Maeda's [J. Sci. Hiroshima Univ. Ser. A. **13**, 11–40 (1944); these Rev. **10**, 546] generalizations are indicated in a footnote; von Neumann's fundamental theorems are proved in chap. 4, and T. Iwamura's theorems [Jap. J. Math. **19**, 57–71 (1944); these Rev. **8**, 35] are proved in chap. 5 by the method of Y. Kawada, K. Higuti and Y. Matusima [Jap. J. Math. **19**, 73–79 (1944); these Rev. **8**, 35]. The continuity of perspectivity, which was first proved directly by Halperin, is proved here indirectly.

Chap. 6 gives the definition and fundamental properties of the regular ring. Chap. 7 shows Maeda's theory of dimension and sub-direct decomposition of regular rings [J. Sci. Hiroshima Univ. Ser. A. **14**, 1–7 (1949); these Rev. **13**, 473], which corresponds to the results in chap. 5. The subsequent four chapters (8. Normalized frames; 9. Matrix rings; 10. Auxiliary rings; 11. Representation theory) are devoted to von Neumann's representation theory of a complemented modular lattice of degree ≥ 4, the proof of which is much simplified by means of K. Kodaira and S. Huruya's method [Zenkoku Shijô Sûgaku Danwakai **168**, **169**, **170** (1938)]. Chap. 12 (the last chapter of the book) deals with orthocomplemented modular lattices (not necessarily finite-dimensional) and ends with Maeda's generalization [J. Sci. Hiroshima Univ. Ser. A. **14**, 93–96 (1950); these Rev. **13**, 425] of G. Birkhoff and von Neumann's representation theorem [Ann. of Math. (2) **37**, 823–843 (1936)] for such lattices.

T. Iwamura (Tokyo).

Calculus of Variations

Boerner, Hermann. Carathéodory's Eingang zur Variationsrechnung. Jber. Deutsch. Math. Verein. **56**, 31–58 (1953).

The author presents a concise and illuminating exposition of Carathéodory's treatment of variational problems. For simple integral problems explicit discussion is limited to the case of the ordinary non-parametric problem in n-space. The concluding sections of the paper give a comparative survey of the various field theories for non-parametric multiple integral problems.

W. T. Reid.

Darbo, Gabriele. L'estremo assoluto per gli integrali su intervallo infinito. Rend. Sem. Mat. Univ. Padova **22**, 399–416 (1953).

The author gives some general conditions that insure the existence of an absolute minimum for an integral of the form $\int_0^\infty f(x, y, y')dx$. Application of this result is made to a case in which the integrand is a polynomial of the second degree in y and y' with coefficients that are continuous functions of x. Finally the author presents some conditions that imply the hypotheses of his preliminary general theorem. The statements of the individual theorems are too complicated to give here.

W. T. Reid (Evanston, Ill.).

Faedo, Sandro. Il calcolo delle variazioni per gli integrali estesi a intervalli infiniti. Ann. Scuola Norm. Super. Pisa (3) **7**, 91–132 (1953).

This paper includes some examples and existence theorems connected with the problem stated in the title. It is a continuation of previous work by the author [see Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **8**, 94–125 (1949); these Rev. **11**, 603].

L. M. Graves (Chicago, Ill.).

Bertolini, Fernando. Studio del funzionale $\int_L \rho^\alpha ds$. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **12**, 90–104 (1953).

The fixed-end-point problem in (x, y, z)-space for the integral $\int (x^2+y^2+z^2)^{\frac{1}{2}\alpha}(\dot{x}^2+\dot{y}^2+\dot{z}^2)^{\frac{1}{2}}dt$ is considered in detail. For $\alpha > -2$ the minimum is attained, while for $\alpha \leqq -2$ this is not always the case. The explicit form of the solution, when it exists, is exhibited. *W. H. Fleming.*

Bertolini, Fernando. Traiettorie luminose in uno spazio trasparente. Esistenza di regioni non illuminate da una sorgente luminosa puntiforme. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 26–28 (1953).

The author discusses the minimum problem for integrals of the form $J(L)=\int_L f(P)ds$, where L lies in a class of "quasi-rectifiable" curves. $f(P)$ is non-negative in a region T from which a finite set of points has been removed and is lower semicontinuous there. A curve is quasi-rectifiable if each of its sub-arcs not containing points of the given finite set is rectifiable. He gives an example of a function $f(P)$, a region A, and a point P_0 lying in $T=E^3$ such that $J(L)$ cannot have a minimum in the class of arcs joining P_0 to $\bar{A}$; here $\{P_0\}$ is the finite set of points excluded from T. If now P_0 is thought of as a point source of light and $f(P)$ the index of refraction, it follows that $\bar{A}$ gets no light. *J. M. Danskin* (Washington, D. C.).

Fite, Wade L. Maximization of return from limited resources. J. Soc. Indust. Appl. Math. **1**, 73–90 (1953).

The author studies the maximization problem for the integral $\int F(x)S(D(x))dx$ subject to $\int D(x)dx = \text{constant}$ and $D(x) \geqq 0$. The function $S(D)$ is taken to be increasing, convex for $D<D_p$ and concave for $D>D_p$. The arguments are heuristic, as the author states. He obtains necessary conditions on the solution, by an argument which he says is "crude and unsatisfactory." Nevertheless the conditions are correct and are the necessary conditions. His work would have been considerably simplified had he taken account of the fact that $S(D)$ can be replaced by a concave function obtained by drawing tangents across the convex portions of the graph. The problem for a concave function is not difficult, and its solution has been published in numerous places, however mostly in literature which is not generally available. The remarks above about reducing the general problem to a concave problem are based on some remarks of L. C. Young [Acta Math. **69**, 229–258 (1938)]. The author applies his results to the problem of how to distribute a given amount of rotenone over a chrysanthemum field in order to achieve the maximum kill of the chrysanthemum aphis. *J. M. Danskin* (Washington, D. C.).

Theory of Probability

Combes, B. Le problème de la généralisation de l'inégalité de Bienaymé. Bull. Trimest. Inst. Actuaires Français **52**, 61–69 (1953).

Let X be a chance variable defined on (a, b) and let $\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x)$ be n functions defined on (a, b). The author gives a geometrical method for determining, for all distributions F on (a, b) with prescribed values for $\int_a^b \varphi_i(x)dF$, $i=1, 2, \cdots, n-1$, upper and lower bounds for $\int_a^b \varphi_n(x)dF$. The same problem is considered when the F's are restricted to certain special classes of distributions. In particular, by making $\varphi_n(x)=0$ for $x \leqq t$ and 1 for $x>t$, this provides a maximum and a minimum for $\Pr\{X>t\}$ among all chance variables X having the same $n-1$ moments and thus provides a generalization of the Tschebyscheff-Bienaymé inequality. *J. L. Snell* (Princeton, N. J.).

Lévy, Paul. Premiers éléments de l'arithmétique des substitutions aléatoires. C. R. Acad. Sci. Paris **237**, 1488–1489 (1953).

A random substitution is defined by assigning probabilities to each of the $n!$ elements of a symmetric group. The group of a random substitution is the smallest group containing all the elements for which the assigned probabilities are not zero. The random substitution obtained by assigning equal probabilities to all elements of this group is said to be the limit substitution, in the Cesàro sense, of the given random substitution. Properties designated by indecomposable, semi-indecomposable, and indefinitely divisible are defined and are said to have almost the same roles as the same terms in the theory of the addition of ordinary random variables. *J. Riordan.*

Weil, Herschel. The distribution of radial error. Ann. Math. Statistics **25**, 168–170 (1954).

An expression suitable for computation is obtained for the probability distribution of $\zeta=(\xi^2+\eta^2)^{\frac{1}{2}}$ where ξ and η are independent Gaussian variables with, in general, unequal means and unequal variances. *Author's summary.*

Korolyuk, V. S. On a problem of B. V. Gnedenko. Dopovidi Akad. Nauk Ukrain. RSR **1951**, 21–23 (1951). (Ukrainian)

The author constructs a distribution function $F(x)$ such that: (1) the symmetrized distribution $F(x)*\{1-F(-x)\}$ is stable of order α, and (2) $F(x)$ does not belong to the domain of attraction of any law. *W. Feller.*

Breny, H. Sur une classe de fonctions aléatoires liées à la loi de Poisson. Bull. Soc. Roy. Sci. Liège **22**, 405–416 (1953).

Assume that the ends of strands of fibre are distributed at random (Poisson process) on the real line and the lengths of individual strands have a distribution $F(l)$, assumed bounded. Let $n(t)$ be the number of strands overlapping the point t on the line. The author calculates the joint distribution of $n(t_1), \cdots, n(t_s)$ for any $t_1, \cdots, t_s$. *J. L. Snell.*

Rios, S. Some probability laws and stochastic processes which reduce to a general Laplace-Stieltjes type. Trabajos Estadistica **4**, 3–10 (1953). (Spanish. English summary)

Apart from the summary this is almost identical with a previous paper by the author [Revista Mat. Hisp.-Amer. (4) **13**, 112–119 (1953); these Rev. **14**, 1101]. *H. P. Mulholland* (Birmingham).

Franckx, Ed. Sur le comportement stochastique des itérations favorables. Bull. Soc. Roy. Sci. Liège **22**, 417–420 (1953).

Let E be an event which occurs with probability p. In a sequence of independent trials, let S_n be the number of runs of length k of the event E in the first n trials. It is proved that $S_n/n \to p^k$ in probability, uniformly in k. A detailed discussion of runs (defined slightly differently than in this paper) is given in Feller's "Probability theory and its applications" [v. 1, Wiley, New York, 1950, chap. 13; these Rev. **12**, 424]. *J. L. Snell* (Princeton, N. J.).

Lukacs, Eugène. Sur une propriété de la loi de Gauss-Laplace. C. R. Acad. Sci. Paris **238**, 444–445 (1954).

Let $X_1, \cdots, X_n$ be independent chance variables with the same Gaussian distribution. Then Daly [Ann. Math. Statistics **17**, 71–74 (1946); these Rev. **7**, 464] showed that, for a certain class of functions $g(x_1, \cdots, x_n)$ having the invariance property $g(x_1, \cdots, x_n)=g(x_1-a, \cdots, x_n-a)$, the chance variables $g(X_1, \cdots, X_n)$ and $\sum_1^n X_j$ are independent. The author proves the following converse to this result. Let $g_s(x_1, \cdots, x_n)$, $s=1, \cdots, n-1$, be $n-1$ functions having the above invariance property and such that the functions of $n-1$ variables defined by $g_s(x_1, \cdots, x_{n-1}, 0)$ have real single-valued measurable inverses. If $X_1, \cdots, X_n$ are independent chance variables such that $g_s(X_1, \cdots, X_n)$, $s=1, \cdots, n-1$, are independent of $\sum_1^n X_j$, then the X_j's have a Gaussian distribution with the same variance.

J. L. Snell (Princeton, N. J.).

Kudô, Akio. On the strong law of large numbers. Mem. Fac. Sci. Kyūsyū Univ. A. **7**, 69–80 (1953).

Let $\{x_k\}$ be a sequence of real-valued random variables with linear regression, i.e.

$$E\{x_m | x_1, \cdots, x_n\}=A^1_{n,m}x_1+\cdots+A^n_{n,m}x_n, \quad m>n.$$

Let $\sigma_{n,m}=E\{x_nx_m\}$ and $M(N)=\sum_{n,m=1}^{N}\sigma_{n,m}$. Then the author proves that if

$$K(N)=M(N)+2\sum_{k=1}^{N}\sum_{n=1}^{k}\left|\sum_{l=k+1}^{N}A^n_{k,l}\right|[M(k)\sigma_{n,n}]^{\frac{1}{2}}$$
$$=O(N^{3-\epsilon}) \quad (\epsilon>0),$$

then $\lim_{N\to\infty} N^{-1}\sum_1^N X_j=0$ with probability one. Special cases where this theorem applies are given and the theorem is used to prove the known result that the law of large numbers holds for a stationary Gaussian Markoff process (not necessarily simple). *J. L. Snell* (Princeton, N. J.).

Dobrušin, R. L. Generalization of Kolmogorov's equations for Markov processes with a finite number of possible states. Mat. Sbornik N.S. **33(75)**, 567–596 (1953). (Russian)

If I is a finite open, semiclosed, or closed interval, let $P(I)$ be an r-dimensional square matrix. If it is supposed that to the null interval ϕ corresponds the identity matrix E, and that $P(I_1 \cup I_2)=P(I_1)P(I_2)$, for disjunct intervals I_1, I_2, then the function P is called multiplicative. An additive function of intervals is defined in the corresponding way, making the null matrix correspond to the null interval. It is supposed in each case that (*) if $I_1 \supset I_2 \supset \cdots$, and if $\bigcap_1^\infty I_n=\phi$, then the function value at I_n goes to that for the null interval. The norm $\|A\|$ of a matrix $A=(a_{ij})$ is defined as $r \max_{i,j}|a_{ij}|$. A matrix function Q of intervals is said to have bounded variation on an interval if $\sup \sum_j \|Q(I_j)-Q(\phi)\|$ (for all decompositions of the interval) is finite. The following results are proved. To each multiplicative [additive] matrix function P [B] of bounded variation on an interval, satisfying (*), there corresponds an additive [a product] integral, defined as a limit of terms of the form $\sum_j\{P(I_j)-E\}$ [$\prod_j\{E+B(I_j)\}$]. The integral defines, on subintervals of the given interval, an additive [a multiplicative] matrix interval function satisfying (*) and of bounded variation. Moreover the inverse operation of either type of integration is the other type. These results are specialized to the case of multiplicative stochastic matrix functions defined in the obvious way by Markov chains, and the corresponding (infinitesimal) additive matrix functions. Under slightly stronger hypotheses (absolutely continuous case) the classical differential equations satisfied by the elements of multiplicative matrix functions are obtained.

J. L. Doob (Urbana, Ill.).

Fortet, R. Les processus stochastiques en cascades. Trabajos Estadistica **4**, 11–34 (1953).

Expository paper.

Vaulot, Emile. Délais d'attente des appels téléphoniques traités au hasard. Ann. Télécommun. **9**, 9–14 (1954).

After a recapitulation of his formulation of the delay problem for telephone calls served at random [C. R. Acad. Sci. Paris **222**, 268–269 (1946); these Rev. **7**, 461], the author devotes his attention to the use of an integral due to Pollaczek [ibid. **222**, 353–355 (1946); these Rev. **7**, 461] for numerical evaluation of the delay distribution (more accurately, its complement). Using the method of steepest descent and thus attending to the longer delays, he gives in detail calculations for two values of time and an occupancy of 0.9, the result of one of which is noted to be in agreement with that of S. O. Rice which has been reported by the reviewer [Bell System Tech. J. **32**, 1266 (1953); these Rev. **15**, 140]. Incidentally some curious integral evaluations are given as obtained by equating the Pollaczek integral and its derivatives with corresponding results known otherwise.

J. Riordan (New York, N. Y.).

Pollaczek, Félix. Sur la théorie stochastique des compteurs électroniques. C. R. Acad. Sci. Paris **238**, 766–768 (1954).

The type II single counter problem [cf., e.g., Feller, Courant Anniversary Volume, Interscience, New York, 1948, pp. 105–115; these Rev. **9**, 294] is generalized to permit an arbitrary distribution for the counter dead-time (supposed constant in the usual treatment), as well as for the inter-arrival times of the events counted. The basic result effectively gives the Laplace-Stieltjes transform of the distribution of times between successive registrations in terms of a function $\psi(-q, 1)$ determined by solution of an integral equation, which with all accompanying definitions is too long to quote. Solutions of the latter are given for the two special cases: (i) exponential inter-arrival distribution, that is, Poisson arrivals, and (ii) exponential dead-time distribution. For the first of these, the generating function for number of registrations in a given time interval which starts with the counter free is also given. [Reviewer's note: it seems likely that these results may be as interesting in the allied field of road traffic problems as in the given setting.]

J. Riordan (New York, N. Y.).

Davis, R. C. On the detection of sure signals in noise. J. Appl. Phys. **25**, 76–82 (1954).

The author applies the classical Neyman-Pearson method to maximizing the probability of detection of a fixed signal in the presence of noise, by means of a finite memory filter. The input is the sum of a fixed signal and noise which is defined by a Gaussian stochastic process with a continuous covariance function. Under specified conditions the optimum filter is that which maximizes the output signal/noise ratio.

J. L. Doob (Urbana, Ill.).

Loeb, Julien. La probabilité des causes dans la technique des télécommunications et des mesures. Ann. Télécommun. **9**, 15–19 (1954).

Mathematical Statistics

Cohen, A. C., Jr., and Woodward, John. Tables of Pearson-Lee-Fisher functions of singly truncated normal distributions. Biometrics **9**, 489–497 (1953).

The authors provide tables for lessening the labor involved in computing estimates of population parameters for singly truncated normal distributions. Let

$$f(x')=\frac{1}{I_0(\xi)(2\pi)^{1/2}\sigma}\exp\left\{-\frac{1}{2}\left(\frac{x'-m}{\sigma}\right)^2\right\},\quad x_0'\leqq x'\leqq\infty,$$

where m and σ are respectively the population mean and standard deviation,

$$\xi=(x_0'-m)/\sigma,\quad I_0(\xi)=\int_\xi^\infty\varphi(t)dt,\quad \varphi(t)=\frac{1}{(2\pi)^{1/2}}e^{-t^2/2},$$

$$Z(\xi)=\varphi(\xi)/I_0(\xi),\quad \frac{m_2(\xi)}{2m_1^2(\xi)}=\frac{1}{2}\left[\frac{1}{Z-\xi}\right]\left[\frac{1}{Z-\xi}-\xi\right],$$

$$W'(\xi)=\frac{1-Z(Z-\xi)}{[1-Z(Z-\xi)][2-\xi(Z-\xi)]-[Z-\xi]^2},$$

$$w'(\xi)=\frac{2-\xi(Z-\xi)}{[1-Z(Z-\xi)][2-\xi(Z-\xi)]-[Z-\xi]^2},$$

and

$$\rho(\sigma,\xi)=\frac{Z-\xi}{[1-Z(Z-\xi)]^{1/2}[2-\xi(Z-\xi)]^{1/2}}.$$

For $\xi=-4(.1)-2.5(.01).50(.1)3$ the authors give $1/(Z-\xi)$ to 8 decimals and $m_3(\xi)/2m_1^3(\xi)$ usually to 8 decimals. For $\xi=-3(.1)2$, $W'(\xi)$ and $w'(\xi)$ are given to 6 decimals and $\rho(\sigma,\xi)$ to 4 decimals. The authors inform the reviewer of the following corrections: for $W'(-2.6)$ read .584 224, $w'(-2.6)$ read 5.328 588. Other corrections to W' and w', if any, will be given in a subsequent issue of Biometrics.

L. A. Aroian (Culver City, Calif.).

Clark, Robert E. Percentage points of the incomplete beta function. J. Amer. Statist. Assoc. **48**, 831–843 (1953).

The table presented here gives to four significant figures the value of $p=P(N, X, \alpha)$ defined by

$$\alpha=\sum_{r=X}^{N}\binom{N}{r}p^r(1-p)^{N-r}=I_p(X, N-X+1)$$

for $N=10(1)50$, $X=1(1)N$, $\alpha=.005$, .010, .025 and 0.50, where $I_p(X, Y)$ is Karl Pearson's incomplete beta function. Applications of the table are indicated. *L. A. Aroian.*

Rao, C. Radhakrishna. Discriminant functions for genetic differentiation and selection. (Part IV of Statistical inference applied to classificatory problems.) Sankhyā **12**, 229–246 (1953).

[For parts I–III see Sankhyā **10**, 229–256 (1950); **11**, 107–116, 257–272 (1951); these Rev. **12**, 511; **13**, 480; **14**, 391.] The author defines familial correlations by generalizing the univariate intraclass correlation to multiple characters. Assuming replicated measurements to be available, he works out the distribution of the characteristic numbers associated with the correlations. This distribution, though obtained formally, offers substantial computational complexity for actual evaluation. The author proceeds to canonical correlation methods, especially those in which the function to be predicted is a specified one instead of that which correlates best. He provides approximate methods for dealing with inequalities in sample size, and formulas for the case in which the population means are replaced by sample means in the function to be predicted.

A. Blake (Linthicum Heights, Md.).

Guttman, Louis. Image theory for the structure of quantitative variates. Psychometrika **18**, 277–296 (1953).

From a "universe of content" consisting of a set of infinitely many quantative statistical variables (not otherwise more clearly defined) let n be arbitrarily selected for study of their joint behavior over a population of subjects by means of linear regression analysis. Common-factor theory has proceeded on the assumption that there exists a basic set of variables, finite in number which are linear combinations of the original variables such that they contain all of the linear relationships among the original variables, that is, there is a finite basis for them as n is increased. There seems to the reviewer to be some confusion here in Guttman's paper—it seems to be implied that all the variables which form a basis are already in those selected for measurement. Otherwise he does not see how it could be expected that the correlation matrix would become diagonal as n is increased.

The author proposes that the linear regression structure of the variables under study be based on the regressions of each variable on the remaining $n-1$. Thus the variable x_j is broken up into p_j+e_j in which p_j, the best least-squares predictor of x_j, is called the image of x_j and e_j, the residual, is called the anti-image of x_j. It is shown that the correlation between e_j and e_k is the negative of the partial correlation between x_j and x_k and that the total correlation between x_j and x_k is the sum of the covariance of p_j and p_k, the parts of x_j and x_k that are in common with the remaining variables, and the covariance of e_j and e_k. This more general situation is compared with the more special one assumed in common-factor theory. Guttman develops the computational procedure for his image analysis which is supposed to disclose as n is increased apart from sampling errors, whether or not the assumptions for common-factor theory are fulfilled. It seems to the reviewer that it would only disclose whether or not a subset of the variables observed would serve as a basis for the complete set. There seems to be considerable vagueness as to what constitutes such a complete set and in practice it does not appear to the reviewer that one would arbitrarily select variables from it.

C. C. Craig (Ann Arbor, Mich.).

Cohen, E. Richard. The basis for the criterion of least squares. Rev. Modern Physics **25**, 709–713 (1953).

Results in this paper are essentially those in F. N. David and J. Neyman [Statist. Res. Mem. London **2**, 105–116 (1938)]. *B. Epstein* (Detroit, Mich.).

Barton, D. E. The probability distribution function of a sum of squares. Trabajos Estadistica **4**, 199–207 (1953). (Spanish summary)

The author gives an expansion for the frequency function of the sum of squares of a finite number of random variables. This expansion has the form of the product of a chi-square frequency function with k degrees of freedom and an infinite series of generalized Laguerre polynomials. *E. Lukacs.*

Plackett, R. L. The truncated Poisson distribution. Biometrics **9**, 485–488 (1953).

The author considers the truncated Poisson distribution $P_r=\lambda^r e^{-\lambda}/r!(1-e^{-\lambda})$, $r=1, 2, 3, \cdots$, in which the zero class

is omitted. He provides a simple unbiased estimate of λ and compares its efficiency with the maximum likelihood estimate of λ. *L. A. Aroian* (Culver City, Calif.).

Rider, Paul R. Truncated Poisson distributions. J. Amer. Statist. Assoc. **48**, 826–830 (1953).

The author provides a method for estimating the parameter of a Poisson distribution which has been truncated such that one or more classes at the lower end are missing. Examples illustrate the method. To the references should be added the paper of Moore [Biometrika **39**, 247–251 (1952); these Rev. **14**, 391] and the paper reviewed above.

L. A. Aroian (Culver City, Calif.).

Bartlett, M. S. Approximate confidence intervals. II. More than one unknown parameter. Biometrika **40**, 306–317 (1953).

Bartlett extends his method [Biometrika **40**, 12–19 (1953); these Rev. **15**, 142] of computing approximate confidence intervals by using the asymptotic normal distribution of the derivative of the log of the likelihood to the case of several parameters. In particular the notion of correcting for the effects of skewness in the distribution is again applied. He also treats the case of nuisance parameters, which involves a deeper and somewhat obscure argument. Applications to the t test, the Behrens-Fisher problem, a problem in the analysis of variance, and to time-series analysis are considered. The relation with goodness-of-fit tests is treated. *H. Chernoff* (Stanford, Calif.).

Gnedenko, B. V., and Studnev, Yu. P. Comparison of the effectiveness of several methods of testing homogeneity of statistical material. Dopovidi Akad. Nauk Ukrain. RSR **1952**, 359–363 (1952). (Ukrainian. Russian summary)

Let $x_1, \cdots, x_n, y_1, \cdots, y_n$ be independent chance variables with the same continuous distribution function (d.f.). Let $S_n(x)$ be the empiric d.f. of the x's, and let $T_n(x)$ be the empiric d.f. of the y's. Define

$$D_n^+ = \sup\,(S_n(x)-T_n(x)), \quad D_n^- = \sup\,(T_n(x)-S_n(x)),$$
$$D_n = \sup\,|S_n(x)-T_n(x)|, \quad R_n = D_n^+ + D_n^-.$$

The authors give without proof: (a) the d.f. of R_n, (b) the correlation coefficient of D_n^+ and D_n^-, (c) a comparison of the variances of D_n^+, D_n, and R_n, (d) the limit of the d.f. of $(\frac{1}{2}n)^{\frac{1}{2}}R_n$, (e) asymptotic formulae for the mean and variance of $(\frac{1}{2}n)^{\frac{1}{2}}D_n^+$, $(\frac{1}{2}n)^{\frac{1}{2}}D_n$, $(\frac{1}{2}n)^{\frac{1}{2}}R_n$, and the correlation coefficient of D_n^+ and D_n^-. A table of some of the quantities described above, for a number of values of n, is included.

J. Wolfowitz (Ithaca, N. Y.).

Rényi, Alfréd. On the theory of ordered samples. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **3**, 467–503 (1953). (Hungarian)

The author's method for studying order statistics is based on the following lemma: Let X_k^* be the kth order statistic of a sample of n observations drawn from an exponential population. Then $X_k^* = \sum_{j=1}^{k} Y_j/(n-j+1)$ where the Y_j are independently and identically distributed random variables which obey the exponential law of the parent population. This property of the exponential distribution was already noticed by B. Epstein and M. Sobel [J. Amer. Statist. Assoc. **48**, 486–502 (1953); these Rev. **15**, 143] in connection with their work on life testing but was not used for a systematic study of order statistics. With the aid of the probability integral transformation, this lemma becomes a convenient tool for the study of the order statistics of a sample drawn from an arbitrary population with continuous cumulative distribution function. A number of known results on order statistics are derived by this method. The method is also used to obtain four theorems which lead to test criteria analogous to the Kolmogorov and Smirnov tests. Let $F(x)$ be the distribution function of the population and $F_n(x)$ the empirical distribution derived from a sample of n. The author's theorems involve the relative deviations $[F_n(x)-F(x)]/F(x)$. As an example, we list two of the simpler results:

$$\text{(a)}\quad \lim_{n\to\infty} P\left\{n^{\frac{1}{2}} \sup_{a\le F(x)\le 1} \left[\frac{F_n(x)-F(x)}{F(x)}\right] < y\right\} = \begin{cases} \Phi(y[a/(1-a)]^{\frac{1}{2}}) - \frac{1}{2} & \text{if } y>0, \\ 0 & \text{if } y\le 0. \end{cases}$$

$$\text{(b)}\quad \lim_{n\to\infty} P\left\{n^{\frac{1}{2}} \sup_{a\le F(x)\le 1} \left|\frac{F_n(x)-F(x)}{F(x)}\right| < y\right\} = \begin{cases} L(y[a/(1-a)]^{\frac{1}{2}}) & \text{if } y>0, \\ 0 & \text{if } y\le 0. \end{cases}$$

Here $a>0$ is an arbitrary positive constant while

$$\Phi(z) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{z} \exp\,[-\tfrac{1}{2}t^2]\,dt,$$

and

$$L(z) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left[\frac{-(2k+1)^2\pi^2}{8z^2}\right].$$

Tables of $L(y[a/(1-a)]^{\frac{1}{2}})$ are given for $y=0.1,\ 0.5(0.5)17$ and $a=0.01(0.01)0.1(0.1)0.5$. Asymptotic probabilities for the suprema of relative deviations over an interval (a, b) with $0<a<b<1$ are also derived. An essential tool in establishing these results are four probability limit theorems which are also of independent interest. These limit theorems are generalizations of results obtained by P. Erdös and M. Kac [Bull. Amer. Math. Soc. **52**, 292–302 (1946); these Rev. **7**, 459]. *E. Lukacs* (Washington, D. C.).

Seal, K. C. On certain extended cases of double sampling. Sankhyā **12**, 357–362 (1953).

The author studies an extended case of the ordinary double sampling procedure where, in addition to the usual two independent samples, there is a third sample available: sample (A) consisting of $y_1, \cdots, y_{n_1}$ of size $n_1 = p_1 n$, sample (B) consisting of $x_1, \cdots, x_k$ and $y_1, \cdots, y_{n_2}$ of size n, and sample (C) consisting of $x_1, \cdots, x_k$ only, of size $n_3 = p_3 n$, where the x_i's have a known multivariate normal distribution and the y's have a univariate normal distribution with known variance but with unknown mean η. Various procedures for estimating η using the three samples (A), (B), and (C) are considered and the optimum selection of sample sizes n and n_3 is discussed. *R. P. Peterson.*

Kitagawa, Tosio. Successive process of statistical inferences. V. Mem. Fac. Sci. Kyūsyū Univ. A. **7**, 81–106 (1953).

[For parts I–IV see these Rev. **13**, 854; **14**, 390, 997.] The author applies his notions of successive processes of statistical inferences to analysis of variance. He then proposes methods of treating the problems of estimation, testing hypotheses and confidence intervals in the light of previous knowledge. His methods seem somewhat arbitrary.

They were constructed as an attempt to relate actual practice in real situations with theory. *H. Chernoff.*

Kitagawa, Tosio. Successive process of statistical inferences. VI. Mem. Fac. Sci. Kyūsyū Univ. A. **8**, 1–29 (1953).

The author discusses prediction problems and treats a sequential case. The next part is a partial preparation to the problem of successive design of experiments. In particular, an extension of the author's previous work on analysis of variance in function space is used.

H. Chernoff (Stanford, Calif.).

Darwin, J. H. Population differences between species growing according to simple birth and death processes. Biometrika **40**, 370–382 (1953).

The author considers various assumptions for stochastic growth of populations or new species to see whether the observed numbers of individuals in a population, or the number and sizes of species in a genus, can thereby be explained. Let $P_n(t)$ be the probability of n individuals in a single population at time t. It is assumed that P_n satisfies equations of the type

$$\frac{dP_0}{dt}=\alpha_1 P_1-\beta_0 P_0,$$

$$\frac{dP_n}{dt}=\alpha_{n+1}P_{n+1}-(\alpha_n+\beta_n)P_n+\beta_{n-1}P_{n-1},\quad n\geqq 1,$$

where α_n and β_n may be functions of t. The simplest result is that if α_n and β_n are independent of t and the population size has an upper bound N, then the binomial distribution $P_n=\binom{N}{n}p^n(1-p)^{N-n}$ can be realized by setting $\alpha_n=\alpha n$, $\beta_n=\beta(N-n)$. Then $p=a+be^{-\mu t}$, where $a=\beta/(\alpha+\beta)$, $\mu=\alpha+\beta$, and b is determined by the initial conditions. Certain other distributions cannot be realized by α_n and β_n independent of time. The stochastic development of new species is treated by assuming that the population develops with birth and death rates, and rate of new speciation, proportional to population size. Comparisons are made with actual data and refinements of the model are discussed.

T. E. Harris (Santa Monica, Calif.).

Moran, P. A. P. The estimation of the parameters of a birth and death process. J. Roy. Statist. Soc. Ser. B. **15**, 241–245 (1953).

The problem considered is reduced to the following random walk: a particle initially at n can move at each step one unit to the right with probability p or one unit to the left with probability $q=1-p$. The particle takes a fixed number, N, of steps, except that it stops if it reaches 0. It is required to estimate p. The maximum likelihood estimate of p is $(x_0-n)/2x_0$ if the particle reaches 0 after x_0 steps, $x_0\leqq N$, and is $(N-n+y_0)/2N$ if the particle is at $y_0\neq 0$ after N steps. The estimate is biased and for suitable but arbitrarily large values of n and N is not normally distributed. The connection with sequential binomial estimation is discussed [Girshick, Mosteller, and Savage, Ann. Math. Statistics **17**, 13–23 (1946); these Rev. **8**, 477]. The distributions of the particle position at time N, if $\neq 0$, and the time at which 0 is reached, if $\leqq N$, are evaluated approximately. The ratio of the death rate to the birth rate is $\mu/\lambda=p^{-1}-1$. In a previous paper [J. Roy. Statist. Soc. Ser. B. **13**, 141–146 (1951); these Rev. **13**, 667] the author discussed the estimation of $\lambda+\mu$. *T. E. Harris.*

Sundrum, R. M. On Lehmann's two-sample test. Ann. Math. Statistics **25**, 139–145 (1954).

This paper considers some properties of a two-sample test, suggested by Lehmann [same Ann. **22**, 165–179 (1951); these Rev. **12**, 726], against general alternatives. Alternative expressions are given for the test statistic; a general formula for the variance is derived and evaluated for the null case; the expectation is obtained in certain nonnull cases; and the exact distributions in the null case are tabulated for some small samples. (Author's summary.)

W. Hoeffding (Chapel Hill, N. C.).

Sundrum, R. M. The power of Wilcoxon's 2-sample test. J. Roy. Statist. Soc. Ser. B. **15**, 246–252 (1953).

A formula for the variance of Wilcoxon's two-sample test under the alternative hypothesis is derived and evaluated for some special cases. The upper bound of this variance is derived, and a population is constructed in which this upper bound is attained. Assuming that the two distributions differ only in the location parameter, the large sample power of the test is compared with that of certain parametric tests in the normal and rectangular cases.

G. E. Noether (Boston, Mass.).

van der Vaart, H. R. An investigation on the power function of Wilcoxon's two sample test if the underlying distributions are not normal. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 438–448 (1953).

Given two populations with continuous densities $f(x)$ and $g(x)=f(x-\mu)$. The power functions of the one- and two-sided Wilcoxon tests of the hypothesis $\mu=0$ are investigated. In particular, formulas are given for the derivatives of the power functions at the point $\mu=0$. It turns out that for unequal sample sizes the two-sided test may be biased if $f(x)$ is skew. For equal sample sizes, the two-sided test is always unbiased. *G. E. Noether* (Boston, Mass.).

Ayant, Yves. L'extension à une variable quantique des notions de fonction de corrélation et de densité spectrale. C. R. Acad. Sci. Paris **238**, 990–992 (1954).

Zadeh, L. A. Theory of filtering. J. Soc. Indust. Appl. Math. **1**, 35–51 (1953).

The purpose of this paper is to outline a unified approach to filter theory. A distinction is made between ideal and optimum filters, the former being defined as filters that achieve a perfect separation of the signal and noise. Three general models of ideal filters are sketched and various criteria of optimum filtering are briefly discussed. Unfortunately some of the detailed statements do not follow from the broad assumptions that are made. *R. Hamming.*

Mazelsky, Bernard. Extension of power spectral methods of generalized harmonic analysis to determine non-Gaussian probability functions of random input disturbances and output responses of linear systems. J. Aeronaut. Sci. **21**, 145–153 (1954).

Applying the methods of generalized harmonic analysis, the author obtains the higher order correlation and spectral functions for time data. The proofs are essentially heuristic. The results can be used to characterize the higher order moments of the data distribution function. In addition the higher order spectral functions can be combined with the transfer function of the linear system to obtain the higher order moments of the output. Finally the conditions under which a non-Gaussian forcing function leads to a Gaussian

output response (again with a linear system) are indicated and discussed. *R. S. Phillips* (New Haven, Conn.).

*Pelegrin, Marc J. **Calcul statistique des systèmes asservis.** Publ. Sci. Tech. Ministère de l'Air, no. 285, Paris, 1953. vi+157 pp. 1200 francs.

This book is primarily a study of statistical servo-mechanism techniques. The author is particularly interested in the application of this theory to the automatic pilot. Chapter I contains a review of the Wiener filter theory which minimizes the mean square error, the calculus of variations leading to a Wiener-Hopf equation. In Chapter II the author considers a modified problem of this sort, namely a servo-mechanism with certain predetermined elements (such as a motor). Allowing other units in the feedback circuit to be arbitrary, the mean square error is again minimized by varying these other elements combined in the form of equivalent filters. Again an equation of the Wiener-Hopf type is obtained and solved. Chapter III consists of a study of relays, that is switching devices. This is a non-linear device and it is attempted to determine the autocorrelation of the output in terms of the statistical properties of the input. In Chapter IV the relay is inserted in the feedback and the overall system is discussed. A final chapter is devoted to the study of atmospheric turbulence. Here the author proposes a method for obtaining a "spectrum of the turbulence". *R. S. Phillips.*

Vowels, R. E. The application of statistical methods to servomechanisms. Australian J. Appl. Sci. **4**, 469–488 (1953).

The paper deals with the optimum mean-square response of a linear system assuming certain subsidiary constraints. Thus in a simple prediction problem, given a message $f_i(t)$ and output of the form $f_0(t)=\int_0^\infty f_i(t-\sigma)h(\sigma)d\sigma$, it is desired to minimize $\lim_{T\to\infty}(2T)^{-1}\int_{-T}^{T}|f_0(t)-f_i(t+\alpha)|^2dt$ assuming, for instance, that $\int_0^\infty h(t)dt=1$. The calculus of variations with Lagrange multipliers leads to an equation of the Wiener-Hopf type which can be solved by the usual technique. The author considers various types of constraints, zero steady-state error (as above), zero "velocity" error, etc. The paper concludes with an application of the method to a simple servo-mechanism. *R. S. Phillips.*

TOPOLOGY

Harary, Frank, and Norman, Robert Z. The dissimilarity characteristic of linear graphs. Proc. Amer. Math. Soc. **5**, 131–135 (1954).

The concept of dissimilarity characteristic, defined by Otter for trees [Ann. of Math. (2) **49**, 583–599 (1948); these Rev. **10**, 53] and extended by the present authors [ibid. **58**, 134–141 (1953); these Rev. **14**, 1113] to Husimi trees, is now extended to general linear graphs. It is again shown to have the value one. *L. Van Hove* (Princeton, N. J.).

Kaluza, Theodor, Jr. Ein Kriterium für das Vorhandensein von Faktoren in beliebigen Graphen. Math. Ann. **126**, 464–465 (1953).

The author considers two operations whereby a connected graph may be changed into another. The first consists in adjoining to a given graph G a path (Weg) of even length or a one-ended infinite path. The path has to have just one vertex, an end, in common with G. In the second operation there is adjoined to G a path of odd length having its two ends, but no other edge or vertex, in common with G.

The author shows that a connected graph G without loops has a factor of degree 1 if and only if it can be obtained from a path of odd length or a one-ended infinite path, by a well-ordered series of operations of the kinds described above. A general loopless graph has a factor of degree 1 if and only if each of its components has this property. Unlike previous criteria for the existence of a factor of degree 1 the author's is not restricted to finite or locally finite graphs. *W. T. Tutte* (Toronto, Ont.).

Nagata, Jun-iti. A characterization of a general uniform space by a system of uniformly continuous functions. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. **4**, 43–49 (1953).

Conditions are given under which two complete uniform spaces R_1, R_2 are biuniformly homeomorphic, and similar conditions if the completeness assumption is dropped. These conditions are in terms of certain partially ordered sets of uniformly continuous mappings of R_1, R_2 into an infinite product of the non-negative real axis by itself. *S. B. Myers* (Ann Arbor, Mich.).

Sakakihara, Kanenji. The structures of neighbourhood systems and the types of convergences. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. **4**, 1–7 (1953).

Starting from (directed) convergence as fundamental, one can define neighborhoods and then a new notion of convergence; starting from neighborhoods, one can define convergence and then new neighborhoods. This paper gives some relations between these notions, and conditions (mostly known) under which the new notions will coincide with the old. One novel feature is that the author restricts his convergent directed systems to be based on a fixed directed set X of suffixes. He also considers what X's are adequate to describe a given topology. *A. H. Stone.*

Padmavally, K. An example of a connected irresolvable Hausdorff space. Duke Math. J. **20**, 513–520 (1953).

All spaces and sets considered are dense-in-themselves. A space R is "irresolvable" (I) if it is not the union of two disjoint dense subsets; R is "strongly irresolvable" (SI) if every subset of R is irresolvable; and R is "maximally irresolvable" (MI) if every dense subset of R is open. Clearly MI$\subset$SI$\subset$I. The author answers a question of E. Hewitt [same J. **10**, 309–333 (1943); these Rev. **5**, 46] by constructing a connected space which is MI (and hence irresolvable) and satisfies the Urysohn (and hence the Hausdorff) separation axiom. He also shows that every SI T_1 space has a finer topology in which it is MI, and that no Hausdorff MI space can be locally connected at any point. A connected Urysohn SI space can be locally connected at one point; it is an open question whether a Hausdorff irresolvable space can be locally connected at all points. *A. H. Stone* (Manchester).

Papy, Georges. Sur les compactifications d'Alexandroff. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 937–941 (1953).

In this note, the author studies one-point compactifications of topological spaces which are not necessarily either Hausdorff or locally compact. *E. Michael.*

Michael, Ernest. Some extension theorems for continuous functions. Pacific J. Math. **3**, 789–806 (1953).

This paper deals with metrizable ANR(Q) and NES(Q) spaces [see Hanner, Ark. Mat. **2**, 315–360 (1952); these Rev. **14**, 396]. As the author remarks, many of the results were known already; most of the new ones have been anticipated by Hanner, in the comprehensive paper referred to. The principal results not in Hanner's paper are as follows. The case Q=perfectly normal is considered; here ANR(Q)$\rightleftarrows$NES(Q)$\rightleftarrows$separable ANR(metric), and similarly for AR(Q) and ES(Q). [For A(N)R(Q) this is implicit in Dowker, ibid. **2**, 307–313 (1952); these Rev. **14**, 396.] Every completely regular A(N)R (completely regular) is (locally) compact. A new example is given, using Hilbert space technique and ideas of Klee and Segal, of a compact Hausdorff ES (metric) which is not an ANR (compact Hausdorff). Finally, the author considers simultaneous extensions of mappings, using a construction given by Dugundji [Pacific J. Math. **1**, 353–367 (1951); these Rev. **13**, 373]. *A. H. Stone* (Manchester).

Krein, S. G. Uniform topology in the space of transformations. Mat. Sbornik N.S. **33(75)**, 627–638 (1953). (Russian)

Denote by $M(Q, R)$ the set of all functions of a set Q into a bicompact Hausdorff space R. Let α be a finite open covering of R. If $\phi_0 \in M(Q, R)$ define $A_\alpha(\phi_0)$ to be the set of all $\phi \in M(Q, R)$ such that, for each $p \in Q$, $\phi(p)$ and $\phi_0(p)$ both lie in an element of α. The collection of neighborhoods $A_\alpha(\phi_0)$ makes $M(Q, R)$ into a uniform space; it is proved to be complete in this uniformity. A subset D of M is defined to be equivariant in case for each finite open covering α of R, there exists a subdivision $Q=\bigcup_1^n E_j$ such that for each $\phi \in D$ and each j, the closure of $\phi(E_j)$ is contained in an element of α. It is proved that D is equivariant if and only if it has a bicompact closure. In case Q is a topological space, let $C(Q, R)$ denote the set of continuous functions from Q to R. Then $C(Q, R)$ is a closed subset of $M(Q, R)$. If $D \subset C$ and ϕ_0 is a limit point of D in the weak topology, a criterion is given for continuity of ϕ_0. This criterion is a generalization of classical theorems. *E. E. Floyd.*

Dedecker, Paul. Quelques aspects de la théorie des structures locales. Bull. Soc. Math. Belgique **1952**, 26–43 (1953).

This is an elaboration of results announced previously [C. R. Acad. Sci. Paris **236**, 771–774 (1953); these Rev. **14**, 1004]. It begins with a recapitulation of the concepts of mathematical structure in the sense of Bourbaki, subordination of structures, and local structures in the sense of Ehresmann [ibid. **234**, 587–589 (1952); these Rev. **13**, 780]. From Ehresmann's concept of pseudogroup of transformations that of paragroup is derived by dropping the axiom about the union of maps. A paragroup γ always has a normalizer, the largest paragroup in which γ is invariant in a certain sense; it consists of the local automorphisms of γ. The connection between pseudogroups, and local structures, defined by an atlas (system of local maps into a given structure) is studied. Among the examples are fiber bundles and leaved manifolds (Reeb). Homology and isomery of local structures s and s' on a set E are defined. For the first, the two structures must give the same topology and must have locally the same pseudogroup of local automorphisms. For the second, to each $x \in E$ there must exist an f in the normalizer of s, leaving x fixed, such that the structure induced by s in the domain (here called source) of f is mapped into the structure induced by s' in the range (here: target) of f. The concept of compatible structures is studied. This is used to define the ($\mathfrak{T}-\mathfrak{S}$)-map of a space with a structure of species ($\mathfrak{T}$) into a space with a structure of species ($\mathfrak{S}$), generalizing the idea, e.g., of differentiable regular map of one manifold into another.

A ($\mathfrak{T}$)-space germ at the point x of a set E with structure of species ($\mathfrak{S}$) is defined to be a maximal collection of ($\mathfrak{T}$)-structures, compatible with the structure on E, on subsets of E, containing x, such that, roughly speaking, any two induce the same structure near x (this implies that the intersection of the two sets is a neighborhood of x on each of the two sets). The set $\mathfrak{E}$ of all such germs has a local structure of species ($\mathfrak{T}$), including a topology (an open set consists of all germs $\mathfrak{E}$, defined by a subset of E with a ($\mathfrak{T}$)-structure, at its various points) and a natural projection onto E. Any ($\mathfrak{T}-\mathfrak{S}$)-map into E can be factored into a ($\mathfrak{T}, \mathfrak{T}$)-map into $\mathfrak{E}$ followed by the projection. A ($\mathfrak{T}$)-subspace of E is an open set of $\mathfrak{E}$ with the induced structure; this generalizes the notion of differentiable submanifold (possibly with multiple points), the submanifold considered as defining a submanifold germ at each one of its points. *H. Samelson* (Princeton, N. J.).

Iséki, Kiyoshi. On a theorem of Reifenberg. Portugaliae Math. **12**, 141–142 (1953).

It is shown that if a finite union of connected sets, with a common point, separates two points on S^2, then the union of some pair of them separates these points [see Reifenberg, Proc. Cambridge Philos. Soc. **49**, 573 (1953); these Rev. **14**, 1109]. *A. D. Wallace* (New Orleans, La.).

Tamura, Takashi. On a generalization of the Jordan-Brouwer-Alexander theorem. Sûgaku **4**, 215–225; **5**, 10–24 (1953). (Japanese)

The main aim of this paper is to generalize the Jordan-Brouwer-Alexander theorem in two senses. As is known, Jordan's theorem concerning a closed polygon in a plane can be proved only under the assumptions of the associative axiom and the axiom of ordering of Hilbert [Grundlagen der Geometrie, 7. Aufl., Springer, Berlin, 1930]. In this paper the space in which Jordan's theorem is proved is generalized as follows. Generalizing Hilbert's two axioms in the arbitrary-dimensional case, the author considers an n-dimensional space R^n, where simplex, complex, and a subdivision theorem are defined. Considering an n-simplex $s(p)$ including a point p in R^n as a neighborhood of p, the space R^n is a Hausdorff space. With this topology the closure $\bar{s}^1$ of a 1-simplex s^1 is assumed to be compact. This is called by the author the "axiom of continuity", because this is equivalent to the axiom of Dedekind concerning $\bar{s}^1$. This topological space R^n is locally compact and is more general than the Euclidean n-space E^n. With respect to a topological image of a sphere in R^n, the Jordan-Brouwer-Alexander theorem is proved in this generalized space R^n. This is the first aim of this paper. Secondly, it is also proved that the same conclusion as Jordan's theorem holds with respect to a set which is more general than the topological sphere. Let P^{n-1} be an $(n-1)$-dimensional orientable closed pseudomanifold [Alexandroff and Hopf, Topologie, Bd I, Springer, Berlin, 1935, p. 193]. Among other conditions (these are denoted by I) for P^{n-1} we assume the following. If two $(n-1)$-simplexes s_1^{n-1}, s_2^{n-1} in P^{n-1} have no point in common, and if the intersection $\overline{\partial s_1^{n-1}} \cap \overline{\partial s_2^{n-1}}$ of the closures of their boundaries ∂s_1^{n-1}, ∂s_2^{n-1} is nonvoid, $\overline{\partial s_1^{n-1} \cap \partial s_2^{n-1}}$ is a common face of s_1^{n-1}, s_2^{n-1} in dimension less than $n-1$.

Instead of this condition it is assumed that at least one point in any $(n-2)$-dimensional simplex s^{n-2} is contained only in the closures of two simplexes with s^{n-2} in common and not contained in any faces of other $(n-1)$-simplexes. (This property is denoted by II.) Consider a topological image of the set with properties I, II in R^n. Then Jordan's theorem holds with respect to the image in R^n.

H. Uehara (Fukuoka).

Livesay, George R. On a theorem of F. J. Dyson. Ann. of Math. (2) **59**, 227–229 (1954).

This note proves the following theorem: Let f be a mapping from the 2-sphere S_2 into a real line E_1. Then there exist two diameters subtending the angle θ for any θ, such that the four end-points are mapped to the same point on E_1. This theorem was proved by F. J. Dyson for the case $\theta=90°$ [Ann. of Math. (2) **54**, 534–536 (1951); these Rev. **13**, 450]. The author makes use of a lemma, Theorem 1, which is also proved in this note, and points out that the Ulam-Borsuk theorem can be proved easily with this lemma. In this field there is a conjecture by Knaster [Colloquium Math. **1**, 30–31 (1947), problem 4]: Let f be a continuous map $S_n \to E_m$. Then given $r=n-m+2$ points $q_1, \cdots, q_r$ on S_n, do there exist $p_1, \cdots, p_r$ on S_n, which are congruent to the $q_1, \cdots, q_r$, and are mapped to a single point in E_m?

H. Yamabe (Princeton, N. J.).

Burger, Ewald. Bemerkungen zu einem Homotopieproblem. Arch. Math. **4**, 470–476 (1953).

Eilenberg has shown that the homotopy classes of maps of an n-dimensional polyhedron $P=|K|$ into a connected space Y whose first $n-1$ homotopy groups vanish $(n \geqq 2)$ stand in one-to-one correspondence with the elements of $H^n(K, \pi_n(Y))$. In fact, if $f: P \to Y$ induces f^*:

$$H^n(Y, \pi_n(Y)) \to H^n(K, \pi_n(Y)),$$

then the element D^n associated with the homotopy class of f is the image f^*B^n of the fundamental class $B^n \in H^n(Y, \pi_n(Y))$.

In the present paper the author studies the precise way in which D^n determines $f^*: H^n(Y, G) \to H^n(K, G)$ for arbitrary coefficient group G. Now $H^n(Y, G) = \mathrm{Hom}\,(H_n(Y), G)$ and, by the universal coefficient theorem,

$$H^n(K, G) = \mathrm{Hom}\,(H_n(K), G) + \mathrm{Ext}\,(H_{n-1}(K), G).$$

It is shown that, if $h^n \in H^n(Y, G)$, then $f^*h^n = h^n \nu D^n$, where ν is the Hurewicz isomorphism $\pi_n(Y) \cong H_n(Y)$. (Note that $h^n\nu$ maps Hom $(H_n(K), \pi_n(Y))$ into Hom $(H_n(K), G)$ and Ext $(H_{n-1}(K), \pi_n(Y))$ into Ext $(H_{n-1}(K), G)$.) By duality and character theory it is possible to use this result to describe the homomorphisms of homology groups induced by f. By a direct argument the author shows that if $h_n \in H_n(K, m)$, $m=0, 2, 3, \cdots$, then $f_*h_n = \nu(D^n \cdot h_n)$, and hence characterizes those systems of homomorphisms ϕ_m: $H_n(K, m) \to H_n(Y, m)$, $m=0, 2, 3, \cdots$, which are induced by maps $P \to Y$.

P. J. Hilton (Cambridge, England).

Serre, Jean-Pierre. Groupes d'homotopie et classes de groupes abéliens. Ann. of Math. (2) **58**, 258–294 (1953).

A non-empty collection $\mathfrak{C}$ of abelian groups is called a "class" if the following three properties hold: (a) every group consisting of only one element belongs to $\mathfrak{C}$; (b) every group isomorphic to a subgroup or quotient group of a group in $\mathfrak{C}$ also belongs to $\mathfrak{C}$; (c) every extension of two groups of $\mathfrak{C}$ again belongs to $\mathfrak{C}$. Examples of classes of groups are the following: all finitely generated groups, all finite groups, all finite groups whose order is divisible only by primes belonging to a given set. A homomorphism $f: A \to B$ of abelian groups is called a "$\mathfrak{C}$-isomorphism onto" if its kernel and co-kernel (i.e., $B/f(A)$) both belong to $\mathfrak{C}$. The author defines various other "$\mathfrak{C}$-notions", which are modifications of the usual notions of the theory of abelian groups obtained by "neglecting" groups belonging to the class $\mathfrak{C}$.

The author's main purpose in introducing classes of abelian groups is to give far reaching generalizations of some standard theorems of homotopy theory. For example, he proves a "Hurewicz Theorem mod $\mathfrak{C}$", as follows: if the space X is arc-wise connected and simply connected, and the homotopy group $\pi_i(X)$ belongs to the class $\mathfrak{C}$ for $i<n$, then the integral homology group $H_i(X)$ belongs to $\mathfrak{C}$ for $0<i<n$, and the natural homomorphism $\pi_n(X) \to H_n(X)$ is a $\mathfrak{C}$-isomorphism onto. There is also a "relative Hurewicz theorem mod $\mathfrak{C}$" for pairs (X, A). As another example, there is a "J. H. C. Whitehead theorem mod $\mathfrak{C}$", of which the following is a particular case: let A and B be connected, simply connected spaces, and let $f: A \to B$ be a continuous map which maps $\pi_2(A)$ onto $\pi_2(B)$. Then the following two statements are equivalent: (a) $f_*: H_i(A) \to H_i(B)$ is a $\mathfrak{C}$-isomorphism onto for all i; (b) $f_*: \pi_i(A) \to \pi_i(B)$ is a $\mathfrak{C}$-isomorphism onto for all i. The proofs of these theorems depend heavily on the general results on the singular homology of fibre spaces given by the author in his thesis [Ann. of Math. (2) **54**, 425–505 (1951); these Rev. **13**, 574].

The author applies these general results to the study of the homotopy groups of spheres, especially in the case where $\mathfrak{C}$ denotes the class of all finite groups whose order is divisible only by certain given primes. Among the more important results thus obtained are the following: (a) if n is even, then $\pi_i(S^n)$ is $\mathfrak{C}$-isomorphic to the direct sum of $\pi_{i-1}(S^{n-1})$ and $\pi_i(S^{2n-1})$, where $\mathfrak{C}$ denotes the class of finite groups whose order is a power of 2; (b) if n is odd, the double suspension E^2: $\pi_i(S^n) \to \pi_{i+2}(S^{n+2})$ is a $\mathfrak{C}$-isomorphism onto for $i<r$, and is $\mathfrak{C}$-onto for $i=r$; here $\mathfrak{C}$ denotes the class of finite groups of order not divisible by a given prime p, and $r=p(n+1)-3$. It follows that the p-primary components of $\pi_i(S^n)$ and $\pi_{i-n+3}(S^3)$ are isomorphic if n is an odd integer $\geqq 3$, and $i<n+4p-6$; (c) the p-primary component of $\pi_{2p}(S^3)$, $\pi_{4p-3}(S^3)$ and $\pi_{4p-2}(S^3)$ are cyclic of order p; for all other integers $i<4p-1$, the p-primary component of $\pi_i(S^3)$ is 0; (d) the author gives neat proofs of the following known facts: $\pi_{n+1}(S^n)$ and $\pi_{n+2}(S^n)$ are cyclic of order 2 for $n>2$; $\pi_3(S^2)$ has 12 elements.

The final chapter of this paper is devoted to miscellaneous applications of the preceding results. The author shows that if K is a finite polyhedron, and x is any element of the integral cohomology group $H^n(K)$, n odd, then there exists any integer $N \neq 0$ and a mapping $f: K \to S^n$ such that $f^*(u) = N \cdot x$, where u denotes the fundamental cohomology class of the n-sphere. The author also studies the homotopy groups of a compact connected semi-simple Lie group, G. According to the Hopf theorem, the rational cohomology ring of G is isomorphic to the rational cohomology ring of a space X which is the product of odd-dimensional spheres. Using the theorem just quoted, the author shows easily that such an isomorphism can be "realized" by a continuous map $f: G \to X$. It follows from the Hurewicz theorem mod $\mathfrak{C}$ that the induced homomorphisms $\pi_i(G) \to \pi_i(X)$ are $\mathfrak{C}$-isomorphisms onto, where $\mathfrak{C}$ denotes the class of all finite groups. The author defines a prime p to be "regular for G" if there exists a continuous map $f: X \to G$ such that f_*: $H_i(X, Z_p) \to H_i(G, Z_p)$ (where Z_p is the ring of integers

mod p) is an isomorphism onto for all dimensions i. The interest in this definition stems from the fact that if the prime p is regular for the group G, and $\mathfrak{C}$ denotes the class of finite groups whose order is prime to p, then the induced homomorphism $f_0: \pi_i(X) \to \pi_i(G)$ is a $\mathfrak{C}$-isomorphism onto. The next theorem asserts that all sufficiently large primes are regular for the group G. In the case of the Classical Lie groups, an explicit condition is given for determining which primes are regular.

The following two errors were pointed out by S. Puppe: (a) in the corollary at the bottom of page 288, it is necessary to suppose that k is the field of rationals; (b) in the first paragraph at the top of page 289 one must assume that $x \in H^n(K, k)$ is contained in the image of $H^n(K, Z)$ in $H^n(K, k)$. *W. S. Massey* (Providence, R. I.).

Moore, John C. Some applications of homology theory to homotopy problems. Ann. of Math. (2) **58**, 325–350 (1953).

In this paper the spectral homology theory associated with fibre spaces is applied to the study of the homotopy groups of spheres and related spaces. Let (E, p, B) be a fibre space with fibre F, let A be a subspace of B and let $D = p^{-1}A$ (all the spaces entering the discussion are assumed arcwise-connected). Then the Serre filtration of the singular chain groups of E [Ann. of Math. (2) **54**, 425–505 (1951); these Rev. **13**, 574] is used to obtain relations between the homology groups of (E, D), (B, A), and F; the fundamental result, relativizing that of Serre, asserts that there exists a spectral sequence $'E_0, 'E_1, \cdots, 'E_\infty$, such that $'E_2$ is isomorphic with $H(B, A; H(F; G))$ and $'E_\infty$ is the graduated group of $H(E, D; G)$, suitably filtered (G is, of course, an arbitrary abelian coefficient group).

The results are first applied, in conjunction with the Cartan-Serre technique of Killing homotopy groups [C. R. Acad. Sci. Paris **234**, 288–290, 393–395 (1952); these Rev. **13**, 675; see also G. W. Whitehead, Proc. Nat. Acad. Sci. U. S. A. **38**, 426–430 (1952); these Rev. **13**, 967], to obtain extensions of the Hurewicz absolute and relative isomorphism theorems. Next, an ingenious technique is described for studying homotopy groups of triads and, in particular, for obtaining the Blakers-Massey triad theorem for an arbitrary principal ideal coefficient domain; this technique consists essentially of replacing an arbitrary excisive triad $(X; A, B)$ by an excisive triad $(\bar{X}; \bar{A}, \bar{B})$ with the same homotopy groups and such that the pair $(\bar{X}, \bar{B})$ is q-connected for all q.

Using the spectral theory developed in the earlier sections, the author is enabled to compute homology groups of certain spaces and apply these computations to the study of homotopy groups of spheres; among the main results is that asserting that, if $n > 1$ is odd and if p is an odd prime, then $\pi_q(S^n) \otimes Z_p = 0$ for $n + 2p - 3 < q < n + 4p - 6$ and $n + 4p - 5 < qn + 6p - 9$. The results of this last section overlap with some of Serre's [C. R. Acad. Sci. Paris **234**, 1340–1342 (1952); these Rev. **13**, 675; and the paper reviewed above] and with some unpublished work of Cartan. [On p. 333, l. 20, a new symbol is needed for the endomorphism $R \to R$ induced by α; β, of course, is not onto.]

P. J. Hilton (Cambridge, England).

Araki, Shôrô. On the triad excision theorem of Blakers and Massey. Nagoya Math. J. **6**, 129–136 (1953).

In this paper a new proof is given of a theorem on triad homotopy groups due to A. L. Blakers and the reviewer [Ann. of Math. (2) **55**, 192–201 (1952); these Rev. **13**, 485]. The main tools used are spaces of paths, and the results of J. P. Serre on the homology of fibre spaces [ibid. **54**, 425–505 (1951); these Rev. **13**, 574]. It should be noted that J. C. Moore has also given a proof of this theorem in the paper reviewed above using the techniques of Serre.

W. S. Massey (Providence, R. I.).

GEOMETRY

Clawson, J. W. A chain of circles associated with the 5-line. Amer. Math. Monthly **61**, 161–166 (1954).

Camier, E. D. Some theorems on conics. Math. Gaz. **38**, 18–25 (1954).

Rodeja F., E. G.-. Note on two formulas for the area of an ellipse. Gaceta Mat. (1) **5**, 159–161 (1953). (Spanish)

Guillotin, M. R. Courbes associées à un quadrilatère. III. Sur une hyperbole dont le centre appartient au cercle des orthopoles de M. Thébault. Bull. Soc. Roy. Sci. Liège **22**, 396–404 (1953).

Lorent, H. Sur certaines chaînes de courbes planes ou de surfaces. Bull. Soc. Roy. Sci. Liège **22**, 456–473 (1953).

Kadeřávek, František. La surface inverse d'une surface de translation cyklo-cyklique et d'une coupolle de Scheffers. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. **1952**, no. 2, 6 pp. (1953). (Czech. French summary)

On traite une surface

$$\eta = z^4 + z^3(2x^2 + 2y^2 - r^2) + z^2(x^4 + y^4 + 2x^2y^2 - 2x^2r^2 - 2y^2r^2) = r^2(x^2 - y^2)^2,$$

qu'on peut obtenir comme une surface inverse d'une surface engendrée par une circonférence invariable mouvée lelong d'une autre circonférence conforme. *Résumé de l'auteur.*

Schuster, Jan. Sur l'hyperboloïde à une nappe. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. **1952**, no. 1, 17 pp. (1953).

Gambier, B. Potentiels circulaires. Faisceaux de cercles; points de Poncelet. J. Math. Pures Appl. (9) **32**, 185–201 (1953).

On appelle potentiel circulaire $(\theta, \epsilon R^2)$ l'expression $M\theta^2 \leqq \epsilon R^2$, ou θ est un point fixe, M un point arbitraire du plan, $\epsilon = \pm 1$. Un faisceau linéaire de potentiels circulaires est

$$[\lambda(M\theta_1^2 - \epsilon_1 R_1^2) + \mu(M\theta_2^2 - \epsilon R_2^2)](\lambda + \mu)^{-1},$$

où le rapport $\lambda : \mu$ peut varier. Un calcul simple montre que c'est égal à $M\theta^2 - \epsilon R^2$ où $\theta\theta_1 : \theta_2\theta = \mu : \lambda$, et

$$\epsilon R^2 = [\lambda\mu\theta_1\theta_2^2 - (\lambda\epsilon_1 R_1^2 + \mu\epsilon_2 R_2^2)](\lambda + \mu)^{-2};$$

l'équation $R = 0$ a toujours des racines réelles pour $\epsilon_1\epsilon_2 = -1$ et pour $\epsilon_1 = \epsilon_2 = -1$; et pour $\epsilon_1 = \epsilon_2 = 1$ quand les cercles (θ_1, R_1) et (θ_2, R_2) n'ont pas de point commun. Dans tous ces cas les points θ' et θ'' correspondants aux deux racines $\lambda : \mu$ sont appelés les points de Poncelet du faisceau; celle-ci peut recevoir la forme $(lM\theta'^2 + mM\theta''^2)(l + m)^{-1}$. L'auteur définit la puissance d'un point M par rapport à un potentiel,

la notion de deux points dits conjugués, la polaire d'un point, l'orthogonalité de deux potentiels et il démontre qu'il y a des faisceaux orthogonaux. Il donne quelques applications sur la géométrie des cercles et celle des coniques en général. *O. Bottema* (Delft).

Weber, Werner. Entbehrliche und unentbehrliche Dreieckstypen beim Hauptsatz über apolare Kurven. Collectanea Math. **5**, 153–174 (1952).

This is a refinement of the author's earlier work on triangles (possibly degenerating so as to have two coincident vertices and two coincident sides) which are inscribed in one conic and self-polar for another [Collectanea Math. **3**, 121–135 (1950); **4**, 71–82 (1951); these Rev. **13**, 153, 968]. *H. S. M. Coxeter* (Toronto, Ont.).

Sydler, J.-P. Sur l'équivalence des polyèdres à dièdres rationnels. Elemente der Math. **8**, 75–79 (1953).

In continuation of a previous paper [Elemente der Math. **7**, 49–53 (1952); these Rev. **14**, 309] the author proves that a polyhedron whose dihedral angles are rational multiples of π is equivalent by decomposition to a polyhedron whose dihedral angles are multiples of $\pi/4$. There seems to be an oversight in the proof on p. 77, where a pyramid *CDEFG* is considered in which the sum of the dihedral angles at the edges *CD*, *CE*, *CF*, *CG* is $<2\pi$. *B. Jessen.*

Shephard, G. C. A construction for Wythoffian polytopes. Canadian J. Math. **6**, 128–134 (1954).

The general kaleidoscope consists of n mirrors (hyperplanes) through a point O in Euclidean n-space, such that every angle between two of the mirrors is a submultiple of π. The images of a point, possibly lying on one or more of the mirrors but equidistant from all the rest, are the vertices of a uniform polytope [G. de B. Robinson, J. London Math. Soc. **6**, 70–75 (1931); H. S. M. Coxeter, Proc. London Math. Soc. (2) **38**, 327–339 (1934)]. The author defines, for the ith mirror p_i, a vector $\mathbf{r}_i = OR_i$ along the line of intersection of the remaining $n-1$ mirrors, of such a magnitude that R_i is at distance $\frac{1}{2}$ from p_i (and consequently at distance 1 from its image in p_i). He remarks that the position vector of the typical vertex of the polytope is $\sum \epsilon_i \mathbf{r}_i$, where each $\epsilon_i = 0$ or 1. In this manner he derives coordinates for all the vertices from equations for the mirrors. He applies a simple modification of the same procedure to the "infinite" kaleidoscope whose $n+1$ mirrors bound a simplex, so that the images of a suitable point are the vertices of a uniform honeycomb. *H. S. M. Coxeter* (Toronto, Ont.).

Herrmann, Horst. Matrizendarstellungen in der Liniengeometrie des P_3. Collectanea Math. **5**, 227–239 (1952).

Remarking that a symmetric polarity or a quadric is represented by a symmetric matrix, and that a line or a linear complex is represented by a skew-symmetric matrix, the author obtains elegant expressions for the common point and common plane of two intersecting lines, for the points where two skew lines meet their transversal from a given point, for the quadric containing the common lines of three linear complexes (one or more special), and for the quadric generated by the polar lines of a given line with respect to a pencil of linear complexes. *H. S. M. Coxeter.*

Szász, Pál. A new presentation of hyperbolic trigonometry by means of the horosphere. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **3**, 521–526 (1953). (Hungarian)

The author derives the basic formulae of hyperbolic trigonometry. He uses spatial considerations and the theorem that the geometry on a horosphere is euclidean. *E. Lukacs* (Washington, D. C.).

Szász, Pál. A new derivation of hyperbolic trigonometry in the plane using classical means. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **3**, 527–533 (1953). (Hungarian)

The author resumes his earlier investigations [Acta. Sci. Math. Szeged **14**, 174–178 (1952); these Rev. **14**, 197] and further simplifies his derivation of trigonometry in the hyperbolic plane. *E. Lukacs* (Washington, D. C.).

Szász, Pál. A direct derivation of hyperbolic trigonometry making use of space. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **3**, 535–559 (1953). (Hungarian)

This is a very detailed and largely expository discussion of hyperbolic trigonometry. Spatial considerations are used. *E. Lukacs* (Washington, D. C.).

Skornyakov, L. A. Projective planes. Amer. Math. Soc. Translation no. **99**, 58 pp. (1953).

Translated from Uspehi Matem. Nauk (N.S.) **6**, no. 6(46), 112–154 (1951); these Rev. **13**, 767.

Schwarz, Gideon. Some theorems on Hudekoff's axioms of orientation. Riveon Lematematika **7**, 13–22 (1954). (Hebrew. English summary)

The author investigates the consistency and independence of a set of axioms presented by N. Hudekoff. [The author's reference to Hudekoff's paper could not be located, but the material may be found in Mat. Sbornik **37**, 169–212 (1930).] These axioms bear the same relationship to the notion of orientation of $(n+1)$tuples of points in n-dimensional Euclidean space as the usual axioms for linearly ordered sets bear to the order in the real line; in fact, they reduce to the latter set of axioms, essentially, when $n=1$. That orientation in Euclidean space is a model for the axiom system is used to prove its consistency. Other models, which satisfy all but one of the axioms, are presented to prove independence. *M. Jerison* (Lafayette, Ind.).

Artzy, Rafael. 4-webs and Moebius' net. Riveon Lematematika **7**, 1–9 (1954). (Hebrew. English summary)

The author considers 4-webs consisting of three families of parallel lines and a pencil of lines through a point. Lines are defined as sets of points satisfying the axioms of incidence, parallelism and order. A net is defined as a set of points and lines belonging to the web and containing all lines passing through pairs of points in the set as well as all points of intersection of lines belonging to the set. In the projective plane the Moebius net is a minimum net. In a 3-web (consisting of three families of parallel lines) where the Brianchon hexagon "closure" postulate (=Schliessungspostulat) holds, the minimum net stands in a one-one correspondence with the lattice of all pairs of integers. In analogy to these results the author derives sufficient (and in a certain respect also necessary) conditions for a minimum net in a 4-web. These conditions are the above mentioned hexagon closure postulate and another closure postulate (too involved to be stated here) which replaces certain sufficient conditions of Desargues type which were derived by the author in a previous paper [thesis, Hebrew Univ., 1945; these Rev. **8**, 343]. The points of the minimum net thus obtained stand in a one-one correspondence with the set of all pairs of rational numbers. Finally the author considers the set of points and lines obtained from a 4-web by adding to it all points and lines generated by rational translations and the line at infinity. He shows that the minimum net in

this case may be characterized by a certain postulate of Desargues type. *J. Levitzki* (Jerusalem).

Artzy, Rafael. On the simplest theorem of incidence in Möbius' net. Riveon Lematematika **7**, 77–78 (1954). (Hebrew. English summary)

By using the methods of the paper reviewed above the author derives a new proof for a theorem due to R. Moufang [Math. Ann. **105**, 536–601 (1931)] to the effect that every theorem of incidence in Moebius' net follows from a certain postulate concerning the complete quadrangle.

J. Levitzki (Jerusalem).

Balasubramanian, N. A theorem on sets of points. Proc. Nat. Inst. Sci. India **19**, 839 (1953).

Theorem. If $p_1, p_2, \cdots, p_n$ are points of the plane, and each circle $C(p_1, p_r, p_s)$ $(r, s=2, 3, \cdots, n; r\neq s)$ contains a fourth point p_t of the set, then the n-tuple is concyclic. The author remarks (as have others) that upon inverting the n-tuple with respect to p_1, the result follows immediately from the Sylvester-Grünwald theorem.

L. M. Blumenthal (Columbia, Mo.).

Convex Domains, Extremal Problems, Integral Geometry

Behrend, F. A. The Steinitz-Gross theorem on sums of vectors. Canadian J. Math. **6**, 108–124 (1954).

Let $\alpha_1, \alpha_2, \cdots, \alpha_p$ be n-dimensional vectors,

$$\sum_1^p \alpha_i = 0, \quad |\alpha_i| \leqq 1,$$

and assume they form a closed polygon $OA_1A_2\cdots A_{p-1}O$ where $OA_1=\alpha_1, \cdots, A_{p-1}O=\alpha_p$, O the origin of n-space. Let (a) $R(\alpha_1, \cdots, \alpha_p)$ be the radius of the smallest circumscribed hypersphere, centered at O; (b) $\bar{R}$ $(\alpha_1, \cdots, \alpha_{p-1})$ the minimum for all possible reorderings of $\alpha_2, \alpha_3, \cdots, \alpha_{p-1}$; (c) c_n the sup of $\bar{R}$ over all p and $\alpha_1, \cdots, \alpha_p$. The results of this paper are: (1) for $n\geqq 3$, $c_n<n$; (2) $c_3\leqq(5+2\sqrt{3})^{\frac{1}{2}}$. These constitute a set of estimates supplementary to those of Steinitz: $c_n\leqq 2(n+1)$ [J. Reine Angew. Math. **143**, 128–175 (1913); **144**, 1–40 (1914)]; Gross: $c_n\leqq 2^n-1$, $c_2=\sqrt{2}$ [Monatsh. Math. Phys. **28**, 221–237 (1917)]; Bergström: $c_n^2\leqq 4c_{n-1}^2+1$ [Abh. Math. Sem. Hamburg. Univ. **8**, 148–152 (1930)]; Bergström: $c_3=\sqrt{2}$ [ibid. **8**, 206–214 (1930)]; Damsteeg and Halperin: $c_n\geqq\frac{1}{2}(n+6)^{\frac{1}{2}}$, $c_2=\sqrt{2}$ [Trans. Roy. Soc. Canada. Sect. III. (3) **44**, 31–35 (1950); these Rev. **12**, 419]. *B. R. Gelbaum* (Minneapolis, Minn.).

Hadwiger, H. Über Gitter und Polyeder. Monatsh. Math. **57**, 246–254 (1953).

Two polyhedra A and B of k-dimensional space R_k are called "Γ-zerlegungsgleich", where Γ is a group of movements in R_k (notation: $A\sim B$) if $A=\sum_1^n A_i$ and $B=\sum_1^n B_i$, where A_i and B_i are polyhedra such that A_i may be carried into B_i by a movement of Γ. They are called "Γ-ergänzungsgleich" if $A+C\sim B+D$, where $C\sim D$. In the case where Γ is a crystallographic group (i.e., a discontinuous group with a finite fundamental domain) the author obtains a simple necessary and sufficient condition for two polyhedra A and B to be Γ-zerlegungsgleich, and shows that if A and B are Γ-ergänzungsgleich, then they are also Γ-zerlegungsgleich.

B. Jessen (Copenhagen).

Hadwiger, H. Additive Funktionale k-dimensionaler Eikörper. II. Arch. Math. **4**, 374–379 (1953).

In part I [Arch. Math. **3**, 470–478 (1952); these Rev. **14**, 1114] the author proved that the class of functionals $\varphi(A)$ defined for all convex bodies A in k-dimensional space, which are additive, continuous, and invariant under motions of A, is spanned by the Minkowski functionals $W_\nu(A)$ $(\nu=0, 1, \cdots, k)$. In the present paper the condition of continuity is replaced by monotony. It is shown that the class of functionals satisfying the modified conditions is the class of linear combinations of the Minkowski functionals with non-negative coefficients. Thus the modified conditions imply continuity. *B. Jessen* (Copenhagen).

Kurita, Minoru. On some formulas about volume and surface area. Nagoya Math. J. **6**, 109–117 (1953).

Using the method of moving frames the author gives a generalization of Guldin's two classical theorems (for volumes and areas) to n-dimensional euclidean and spherical spaces. [Two items could be added to the bibliography: Bloch and Guillaumin, La géométrie intégrale du contour gauche, Gauthier-Villars, Paris, 1949; these Rev. **10**, 569; Pöttker, Arch. Math. **2**, 192–198 (1950); these Rev. **12**, 44.] The same method leads to the following generalization of Holditch's theorem to n-dimensional euclidean space: Let P_i $(i=0, 1, \cdots, n)$ be $n+1$ points on a straight line which is moving in such a way that the points P_i describe closed hypersurfaces enclosing solids of volumes V_i respectively. Then the following formula holds

$$\sum_{i=0}^{n} V_i/(d_{01}d_{12}\cdots d_{i-1,i}d_{i+1,i}\cdots d_{n,i}) = (-1)^n kI_n,$$

where d_{ij} are the oriented distances from P_i to P_j, I_n = volume of unit n-sphere and k is an integer which depends upon the motion. The generalization to the spherical case is only sketched [for $n=2$ see E. Vidal Abascal, Revista Mat. Hisp.-Amer. (4) **7**, 132–142 (1947); these Rev. **9**, 608].

L. A. Santaló (Buenos Aires).

Algebraic Geometry

Spampinato, Nicolò. Le varietà dell' S_3 determinate da una coppia ordinata di curve algebriche del piano complesso. Ricerca, Napoli **4**, no. 3–4, 3–11 (1953).

Fadini, Angelo. Studio di una trasformazione cremoniana dell'S_3 complesso dedotta da una trasformazione quadratica dell'S_3 triduale con due punti eccezionali coincidenti. Ricerca, Napoli **4**, no. 3–4, 12–20 (1953).

Godeaux, Lucien. Remarque sur la surface de Kummer. Bull. Soc. Roy. Sci. Liège **22**, 368–373 (1953).

Turri, Tullio. Sulle funzioni algebriche generali di due variabili colla stessa curva di diramazione e birazionalmente distinte. Rend. Sem. Fac. Sci. Univ. Cagliari **23**, 9–15 (1953).

Derwidué, L. Sur la réduction des singularités d'une variété algébrique. Mém. Soc. Roy. Sci. Liège (4) **13**, no. 1–2, 1–41 (1953).

The introduction to this memoir begins by recalling the author's previous memoir on the same subject [Math. Ann.

123, 302–330 (1951); these Rev. 13, 67], and it records the fact that that memoir was founded on a property of first polars which was later found to be false. Since then, as the author says, two criticisms of the memoir in question have been published, of which one was embodied in Segre's memoir [Ann. Mat. Pura Appl. (4) 33, 5–48 (1952); these Rev. 14, 683], while the other was contained in Zariski's review [these Rev. 13, 67]. The author says that in the present memoir he proposes to extract from the above criticisms those elements which bring constructive contributions to the problem and to show that thanks to such contributions it is possible to transform his original reasoning in such a way as to make it attain its object.

In the present work Derwidué, in fact, follows Segre closely for all the early part. First he brings his own use of "elementary transformations" into complete agreement with Segre's use of "dilatations"; then he formulates carefully and in detail the basic mechanism of reduction—the conditions under which a minimum reduction of intersection multiplicity by dilatation can be guaranteed; and finally he accepts, with obvious enthusiasm, Segre's new concept of "associated behaviour", this being for him an ideal means of retrieving the failure of first polars to play the part he had originally assigned to them. From here on, however, Derwidué claims to have made certain new and vital simplifications which enable him to by-pass much of the detailed analysis contained in Segre's work on surfaces, and which enable him ultimately to solve the general problem completely. In these developments his ideas on associated behaviour are based on Segre's first account which is contained in the paper cited above. A somewhat different aspect of the same subject was later published by Segre [Rend. Circ. Mat. Palermo (2) 1, 373–379 (1953); these Rev. 15, 351].

We now try to summarize Derwidué's general plan. Let U be a non-singular manifold of dimension $v+1$, and let this contain the manifold V, of dimension v, which is to be desingularized by elementary transformations of U (and its successive transforms). If m is the multiplicity maximum on V, we have to reduce m. Let $[W]$ be the system of "$(m-1)$th polars of V on U", i.e., the system of manifolds W, of dimension v, in which U is met by the ordinary $(m-1)$th polar primals of all points of the ambient space S_N of U with respect to all the primal cones in S_N which project V. Every W (as the author leaves us to verify) has associated behaviour of index $m-1$ (in Segre's sense) with V at every m-ple point of V; and for every m-ple point of V there exist members of $[W]$ whose association with V at this point is regular. The W therefore pass simply "in general" through all the (proper) m-ple points of V; and it appears, by known properties of associated behaviour and by a new theorem of Derwidué's to which we shall refer, that the W also pass (in the usual conventional sense) through all possible improper m-ple points of V that could be realized by sequences of dilatations applied to V. Now let $|w|$ be the system of $(v-1)$-folds which the W cut on V residually to the m-ple $(v-1)$-folds of V. The three stages of Derwidué's reduction are then as follows: (1) proceed with eliminating the base of $|w|$ until the stage is reached at which the image $|w^*|$ of $|w|$ has no base points which are m-ple for V^* (the corresponding transform of V); show that all the m-ple points of V^* then belong to a finite number of $(v-1)$-folds which may be singular and may meet each other; (2) reduce the singularities of the m-ple $(v-1)$-folds of V^* (with the obvious inductive hypothesis) and show that this can introduce on the final model V^{**} only a finite number of non-singular m-ple $(v-1)$-folds; (3) eliminate the m-ple $(v-1)$-folds of V^{**}, and show that this does not introduce any new m-ple point on the final model V^{***}.

We attempt to indicate now, in connection with stage (1) above, why Derwidué expects that a $|w^*|$ with no base points that are m-ple for V^* will ensure the absence from V^* of any m-ple loci except those of dimension $v-1$. Suppose that we are dilating any non-singular m-ple manifold γ of V (or of any of its subsequent transforms). Then γ is a (generally) simple base manifold of $[W]$, and it gives rise, by the dilatation, to a manifold γ', of dimension $v-1$, on the transform V' of V; also γ' may have any multiplicity m' $(1 \leqq m' \leqq m)$ on V' and it may or may not be a base manifold of $[W']$ (according to whether γ is or is not "unodal" on V). If $m'=m$, γ' is m-ple on V and hence a base manifold of $[W']$ (since the W' are associated to V'). If $m'<m$ and γ' is not a base manifold of $[W']$, then any point of γ' that is (exceptionally) m-ple for V' is necessarily a base point not only of $[W']$ but also of $[w']$. But if $m'<m$ and γ' is a base manifold of $[W']$, it is by no means clear what may happen about special points of γ' that may be m-ple on V'. The crucial result obtained by Derwidué, however, is that in this last case every m-ple point of V' on γ' is either (i) a point of intersection of γ' with an m-ple $(v-1)$-fold of V', or (ii) a base point of $[w']$. Such points, therefore, cannot escape from the machine, and this leads to the desired result. This same theorem is needed again and again at every stage. Its proof, incidentally, which is based on a special result in Segre's first account of associated behaviour, is rather delicate, and the reviewer would like to see some indication of an algebraic confirmation.

So much for the general plan. As applied first to the case of a surface the detailed working out of the plan is completed in six pages; and as applied next to a threefold the detailed working out occupies eight pages. Thereafter, having shown that similar methods will deal inductively with the general case, Derwidué claims the complete solution of the problem. Furthermore, in an appendix, he claims to have established, by a method of descent, the more general result: If V is any irreducible v-fold immersed in a non-singular $(v+r)$-fold U $(r \geqq 1)$, then V can always be desingularized by a finite succession of elementary transformations operating on U and its successive transforms.

The question of the substantial validity of the methods and conclusions of this paper is one on which the reviewer is at present unwilling to express any opinion. The whole investigation, in its present amended form, is certainly interesting, and it is presented with a skill of exposition which greatly adds to its attraction. On the other hand, the reviewer has two important general reservations on his mind. First, the whole investigation is completely tied up with Segre's new theory of associated behaviour, and the reviewer feels for his own part that the deeper implications of this theory may not as yet be sufficiently understood to warrant such unlimited confidence in applying its results as Derwidué seems to possess. Secondly, Derwidué's idiom is uncompromisingly geometric; and this means that very often his more intricate arguments have to be analysed step-by-step so that their formal structure and connections can be recognized. Thus, for example, the reader has to find his own interpretation in p. 23 for the sentence, "We are then led to define on the generic W a sufficiently general system of curves ω which have at A regular $(\mu-1)$-associated behaviour with respect to the corresponding w"; the more so, in this instance, because a few lines later we have to combine an infinity of such systems (each "sufficiently

ample") into a single system and then to consider the generic curve of the latter (its irreducibility being presumed). In this instance, actually, it would seem that the $(\mu-1)$th polars of w on W should do; but this again may remind us that we have still to verify the really fundamental result that on any non-singular $(\nu+1)$-fold U, the ith polar ν-folds of a given ν-fold V are everywhere i-associated, and in general regularly i-associated, to V in Segre's sense of the term.

We remark only, in conclusion, that Derwidué claims for reductions by elementary transformations alone—because of their exceptional simplicity—a very particular interest; and he considers therefore that Zariski's review reference to his previous extensive use of general monoidal transformations (of which elementary transformations are a very special case) cannot be regarded as very relevant

J. G. Semple (London).

Snapper, Ernst. Equivalence relations in algebraic geometry. Bull. Amer. Math. Soc. **60**, 1–19 (1954).

Expository lecture on the following topics: cycle groups; general discussion of equivalence relations among cycles; algebraic systems and algebraic equivalence; linear series and linear equivalence; rational equivalence; advice to beginners in Algebraic Geometry. Many simple illustrative examples are given. *P. Samuel* (Clermont-Ferrand).

Differential Geometry

Gheorghiev, Gh. Sur quelques généralisations du théorème de Réaumur pour les courbes d'une surface. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **4**, 497–503 (1952). (Romanian. Russian and French summaries)

Hartman, Philip, and Wintner, Aurel. On parabolic curves on surfaces. Amer. J. Math. **76**, 81–86 (1954).

The theorem proved here is a modification of one due to P. Franklin [J. Math. Physics **13**, 253–260 (1934)]. Let $S(x, y)$ be a surface of class C^2 defined over an open (x, y)-domain, D. Suppose that there exists in D a Jordan arc of class C^1 which divides D into two domains D^+ and D^- in which the curvature $K>0$, $K<0$ respectively. The image of this arc on S is a parabolic arc A. Suppose further that at any point P of A the tangent plane $T(P)$ of S at P is a supporting plane of S near P. Then A is a plane curve and T is independent of P. *C. B. Allendoerfer.*

Wunderlich, Walter. Zur Geometrie der Drehflächen und ihrer geodätischen Linien. Monatsh. Math. **57**, 199–216 (1953).

Consider any surface of revolution Φ_1 and on the surface a one-parameter family of geodesics, congruent to each other under rotation. The congruence of tangents to these geodesics are tangent to a second surface Ψ_1 which is also tangent to the osculating planes of these geodesics. Since there is a one-parameter family of such sets of geodesics, there corresponds to Φ_1 a one-parameter family Ψ of these director surfaces (Leitflächen). It is then established that the given surface Φ_1 can be embedded in a unique one-parameter family of surfaces of rotation Φ such that the pair of families (Φ, Ψ) have the property that each family consists of the director surfaces of any surface of the other family.

If a family of surfaces is projected orthogonally on a plane, the boundaries of the resulting point sets may be called the boundary curves (Umrisslinien). The pair of families (Φ, Ψ) is shown to have the property that, when these surfaces are projected orthogonally on any plane not perpendicular to the axis of revolution, the system of boundary curves is an orthogonal net, and is always similar to the net of intersections with any meridian plane.

At any point P of a surface Φ_1 of family Φ the normal line and tangent plane meet the axis of rotation in points U and V respectively. As P varies over Φ_1 this induces a point correspondence $U \rightarrow V$ on the axis. It is shown that the point correspondence completely determines both families Φ and Ψ. Analytic determination of the families from the correspondence is discussed, with special emphasis on the case when the correspondence is projective.

S. B. Jackson (College Park, Md.).

Bol, Gerrit. Über die Flächen, bei denen die Diagonalen der Vierseite von Demoulin linearen Kongruenzen angehören. Math. Z. **59**, 97–150 (1953).

It is known that by the representation of Klein both u and v tangent congruences of a surface $X(u, v)$ with asymptotic parameters u, v in a three-dimensional projective space R_3 correspond to two surfaces $U(u, v)$ and $V(u, v)$ in a five-dimensional linear space S_5. The Laplace sequence

$$\cdots, V_{-2}, V_{-1}, V=V_0, U=U_0, U_1, U_2, \cdots$$

generated by U, V is called the Godeaux sequence of $X(u, v)$. A surface is called a congruence surface if the diagonals of its quadrilaterals of Demoulin belong to a linear congruence. The purpose of this paper is to study whether the Godeaux sequence of a congruence surface can be closed. The answer to this problem is affirmative and the period of the sequence must be $4n \geq 8$.

It is known that if the Godeaux sequence of a surface S is closed with a period $4n$, then the two surfaces U_{2n-1} and U_{2n} in the sequence are the representations in S_3 of the two tangent congruences of another surface in R_3 called the opposite surface of S. If the period of the Godeaux sequence of a congruence surface is divisible by 8, then the surface and its opposite surface either are projectively equivalent or correspond to each other in the null system of a linear complex. The transformation from a surface S in R_3 to any enveloping surface of the Lie quadrics of S is called an M-transformation, and this definition is reversible for a projective minimal surface. Repeated applications of M-transformations lead from a projective minimal surface to a system of projective minimal surfaces, which is endless in both directions and called an M-system. By means of M-transformations and the representation in S_3 of the linear congruence to which belong the quadrilaterals of Demoulin of a congruence surface, the author defines twenty-four classes of real congruence surfaces in seven of which all surfaces have closed Godeaux sequences. The opposite surface of a special congruence surface S, called a hyperbolic curved surface, belongs to an M-system if S has a closed Godeaux sequence. The problem whether this is also true for another special congruence surface, called an elliptic curved surface, is still unsolved.

In the above twenty-four classes most surfaces are projective surfaces of revolution, and in particular four of them are ordinary surfaces of revolution whose meridians depend on two periodic functions. Finally, a simple example is given for a class of coincidence surfaces which are special W-surfaces and tetrahedron surfaces at the same time.

Each surface of the equation

$$z=x^{\tan A}y^{\tan B},\quad A+B=\pi/4,$$

is a congruence surface, whose Godeaux sequence is closed with a period $4n$ if and only if A is an integral multiple of $\pi/4n$. A special one of such surfaces for $n=2$ and $A=\pi/8$ was obtained by M. Carton. *C. C. Hsiung.*

Pinl, Max. Über einen Satz von G. Ricci-Curbastro und die Gausssche Krümmung der Minimalflächen. Arch. Math. **4**, 369–373 (1953).

If ds^2 denotes the first fundamental form for a surface S in Euclidean 3-space and K its curvature, a theorem of G. Ricci-Curbastro [W. Blaschke, Einführung in die Differentialgeometrie, Springer, Berlin, 1950, §72, p. 124; these Rev. **13**, 274] states that a necessary and sufficient condition that S be minimal is that the quadratic form $ds^2=\sqrt{(-K)}ds^2$ shall have zero curvature. Since both the hypothesis and conclusion of this theorem are meaningful, independent of the embedding in 3-space, the author here investigates the question of the validity of the theorem in Euclidean n-space. It turns out that the theorem cannot be thus extended. It is shown that the theorem, in n-space, is equivalent to the existence of a coordinate system on a minimal surface for which K takes on the Study normal form. An example in 4-space shows that this is not always possible.

S. B. Jackson (College Park, Md.).

Pinl, M. B-Kugelbilder reeller Minimalflächen in R_4. Math. Z. **59**, 290–295 (1953).

For two-dimensional surfaces in Euclidean 4-space Blaschke [Ann. Mat. Pura Appl. (4) **28**, 205–209 (1949); these Rev. **11**, 741] has obtained a pair of spherical maps, called B-spherical maps. It is here established that these spherical maps are conformal in the case of minimal surfaces. This is analogous to the fact that the usual Gaussian spherical map of surfaces in 3-space is conformal for minimal surfaces. *S. B. Jackson* (College Park, Md.).

Ohmann, D. Einige dem Vierscheitelsatz verwandte Sätze über Eilinien. J. Reine Angew. Math. **192**, 74–76 (1953).

If $h(\theta)$ is the supporting function of an oval, a linear aggregate of the supporting function is defined as a function having the form of the following finite double sum $y(\theta)=\sum_{k,l}a_{kl}h^{(k)}(\theta+\alpha_l)$, where the a_{kl} and the α_l are given constants, or a finite sum of integrals of the form $y(\theta)=\sum_k\int_0^{2\pi}a_k(\lambda)h^{(k)}(\theta+\lambda)d\lambda$, where the $a_k(\lambda)$ are given functions of λ. If the linear aggregate $y(\theta)$ is continuous and invariant under translation, it is established that $y(\theta)$ takes on its mean value at least four times. If, moreover, $y(\theta)+y(\theta+\pi)=\text{const.}$, then $y(\theta)$ takes on its mean value at least six times.

Taking the case when y is the radius of curvature, namely $h+h''$, this yields the Four-vertex Theorem. Considering other linear aggregates the author obtains a sequence of theorems which are therefore analogues of the Four-vertex Theorem. Section 2 of the paper contains two theorems concerning polygons with fixed vertex angles circumscribed about the oval. In section 3 are included theorems on subarcs of the oval having fixed total curvature, while section 4 generalizes these theorems to pairs of ovals. A representative theorem is the following: Among all polygons with prescribed vertex angles α_k circumscribed about an oval (of constant breadth), there are at least four (six) whose perimeter equals $(L/\pi)\sum\tan(\frac{1}{2}\alpha_k)$, where L is the length of the oval. It should be noted that, as printed in the paper, the factor $1/\pi$ in the mean perimeter mentioned in the theorem is inadvertantly omitted. *S. B. Jackson.*

Matsumura, Soji. Über Flächen und Kurven. LXI. J. Osaka Inst. Sci. Tech. Part I. **2**, 41–42 (1950).

Matsumura, Soji. Beiträge zur Geometrie der Kreise und Kugeln. LXI. J. Osaka Inst. Sci. Tech. Part I. **2**, 47–50 (1950).

Beatty, S. On ruled surfaces. Amer. Math. Monthly **61**, 176–179 (1954).

Jonas, Hans. Die Differentialgleichung der Affinsphären in einer neuen Gestalt. Math. Nachr. **10**, 331–352 (1953).

Verf. erhält die folgende Darstellung der Affinsphären: $x=R$, $y=Ru$, $z=-v^{-1}+R\cdot(\omega u-\phi)$, wo die Funktion $R(u,v)$ eine Lösung der Differentialgleichung

$$(R^{-2}R_uv^{-2})_u=(R^2R_vv^{-2})_v$$

ist und die Funktionen ω und ϕ mittels Quadraturen aus R bestimmt werden können. Dies führt zu einer neuen Behandlungsweise der mit Affinsphären zusammenhängenden Probleme. Es zeigt sich dass die Asymptotenlinien sich mittels Quadraturen bestimmen lassen. Als Spezialfälle werden die geradlinigen Affinsphären und die affinsphärischen Rotationsflächen besprochen. Zum Schluss werden die Affinsphären im zusammengesetzten W-Verbände untersucht. *J. Haantjes* (Leiden).

***Kuiper, N. H. Sur l'immersion isométrique.** Colloque de topologie et géométrie différentielle, Strasbourg, 1952, no. 4, 3 pp. La Bibliothèque Nationale et Universitaire de Strasbourg, 1953.

This note is a sequel to the paper by Chern and Kuiper [Ann. of Math. (2) **56**, 422–430 (1952); these Rev. **14**, 408]. The following algebraic lemma is proved: Let

$$1\leqq i,j,k,m\leqq n;\quad n+1\leqq r,s,t\leqq n+N;$$
$$p^{ij}=x^iy^j-x^jy^i;\quad A_{rij}=A_{rji};$$
$$F(p)=\sum_{r=n+1}^{n+N}(A_{rik}A_{rjm}-A_{rim}A_{rjk})p^{ij}p^{km}$$

and assume that $F(p)$ is a negative definite form in p^{ij}. Then $N\geqq\min(3,n-1)$. The lemma is used to prove that a V_5 (which is the metric product of a U_3 of non-positive sectional curvature and a U_2) can not be imbedded in a sphere S_7. Furthermore, a V_5 (which is the metric product of a compact U_3 having non-positive sectional curvature and a compact U_2) can not be imbedded in Euclidean E_4. *C. B. Allendoerfer* (Seattle, Wash.).

Rembs, Eduard. Zur Verbiegbarkeit konvexer Kalotten. Arch. Math. **4**, 366–368 (1953).

This is a third proof of the author's result that certain convex disks are rigid under certain infinitesimal deformations [Math. Z. **56**, 271–279 (1952); these Rev. **14**, 901; also Grotemeyer, ibid. **58**, 272–280 (1953); these Rev. **15**, 158]. *P. Scherk* (Saskatoon, Sask.).

Castoldi, Luigi. Significato geometrico del parametro caratteristico sopra una geodetica di lunghezza nulla in una varietà Riemanniana con metrica totalmente iperbolica. Atti Accad. Ligure **8** (1951), 123–130 (1952).

The author gives an interpretation of the parameter on null-geodesics for which $\delta_1(dx^h/dt)=0$ by means of the trace of Γ on a family of parallel hypersurfaces. *J. Haantjes.*

***Gyarmathi, László. Konstruktive Lösung der metrischen Aufgaben des vierdimensionalen linearen Raumes auf Grund der Maurinschen Abbildung.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 653–664. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian and German summaries)

Verf. löst, bei Zugrundelegung des im Titel der Arbeit angegebenen Maurinschen Abbildungsverfahrens, die folgenden metrischen Fundamentalaufgaben. Bestimmung der wahren Grösse einer Strecke und derjenigen eines Winkels. Weiter wird die Konstruktion von zueinander orthogonalen, dreidimensionalen Räumen angeführt. Im dreidimensionalen Raume können die beiden ersten Aufgaben, im Falle des Mongeschen Abbildungsverfahrens, durch Einführung einer neuen Rissebene, bzw. Drehung um eine Achse, bewerkstelligt werden. Durch eine geschickte Verallgemeinerung dieses Verfahrens gelingt Verf. die Lösung der beiden erstgenannten Aufgaben. Bei der Konstruktion von zueinander orthogonalen Räumen macht Verf. von dem Umstand Gebrauch, dass die zu diesen Räumen gehörigen unendlich fernen Elemente einander in derjenigen Polarität entsprechen, deren Fundamentalgebilde die absolute Kugel des R_4 ist. *O. Varga* (Debrecen).

Libermann, Paulette. Forme canonique d'une forme différentielle extérieure quadratique fermée. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 846–850 (1953).

Let Ω be a closed exterior differential 2-form of rank $2n$ in a differentiable manifold V_{2n}. Under what assumptions can Ω be reduced to the canonical form

$$dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n?$$

It is shown that this is possible if V_{2n} is of class C'', and Ω of class C' (i.e. its coefficients with respect to any coordinate system of V_{2n} are of class C'). The proof makes use of the notions of adjoint form with respect to Ω, and of effective form, as introduced by the author [Thèse, Strasbourg, 1953, to appear in Ann. Mat. Pura Appl.], of E. Cartan's reduction methods for Pfaffians, and of Frobenius' theorem on completely integrable systems. *H. Samelson.*

Ispas, C. I. Les identités de Veblen dans les espaces généralisés. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **4**, 533–539 (1952). (Romanian. Russian and French summaries)

Dans cette note, on établit les identités de Veblen, dans les espaces généralisés, sous la forme:

$$V^q_{[i\cdot km]i} = L^q_{k\cdot[m]i} + L^q_{m\cdot k[i]} + L^q_{i\cdot m[i]k} + L^q_{[i\cdot k]m} = 0,$$

où les $L^q_{k\cdot[m]i}$ sont donnés par

$$L^q_{k\cdot[m]i} = K^q_{k\cdot[m]i} - \frac{\partial K^q_{k\cdot[m]}}{\partial x^i} + x'^i \frac{\partial K^q_{k\cdot[m]}}{\partial x'^p} \Gamma^p_{ji} - K^p_{k\cdot[m]} \Gamma^q_{pi} - K^q_{p\cdot k[i]} \Gamma^p_{m]i},$$

dans ces formules $K^q_{k\cdot lm}$ étant le tenseur fondamental de courbure de L. Berwald. On observe que les identités de Veblen dans les espace généralisés conservent la symétrie de ceux du cadre de la connexion affine. (Author's summary.)

V. Hlavatý (Bloomington, Ind.).

Nasu, Yasuo. On the projectively connected spaces with homogeneous coordinates whose groups of holonomy fix a hyperquadric. Kumamoto J. Sci. Ser. A. **1**, no. 2, 1–7 (1953).

Verf. betrachtet eine Mannigfaltigkeit mit homogenen Koordinaten $x^0, x^1, \cdots, x^n$ und einer projektiven Übertragung $\Pi^\alpha_{\mu\nu}$ entsprechend der Theorie von Schouten, van Danzig und Haantjes. Die zu dieser Mannigfaltigkeit gehörige Holonomiegruppe möge eine, in einem Tangentialraum des Punktes P liegende nicht ausgeartete Hyperfläche Q_{n-1}, die P nicht enthält, invariant lassen. Unter diesen Bedingungen zeigt Verf., dass sich eine solche affine Übertragung aus den $\Pi^\alpha_{\mu\nu}$ herleiten lässt, deren Parameter Γ^i_{jk} die Christoffelschen Symbole eines Tensors zweiter Stufe sind. Dieser Tensor ist durch die Hyperfläche Q_{n-1} bestimmt und bildet den metrischen Tensor eines Einsteinschen Raumes von konstanter, nicht verschwindender Krümmung.

O. Varga (Debrecen).

Nasu, Yasuo. On normal coordinates in projectively connected spaces with homogeneous coordinates. Kumamoto J. Sci. Ser. A. **1**, no. 2, 8–13 (1953).

Zugrunde gelegt sei eine n-dimensionale Mannigfaltigkeit mit homogenen van Danzigschen Koordinaten $x^0, x^1, \cdots, x^n$. In derselben sei nach Schouten und Haantjes eine projektive Übertragung durch geometrische Objekte $\Pi^\lambda_{\mu\nu}$ definiert. Verf. zeigt, dass sich in einer solchen Mannigfaltigkeit, in einem beliebigen Punkt $\mathfrak{x}^\lambda$ Normalkoordinaten einführen lassen, die dadurch definiert sind, dass die Relationen

$$(\Pi^\lambda_{(\mu\nu)})_{\mathfrak{x}} = (\partial_{(\alpha}\Pi^\lambda_{\mu\nu)})_{\mathfrak{x}} = (\partial_{(\alpha}\partial_\beta\Pi^\lambda_{\mu\nu)})_{\mathfrak{x}} = \cdots = 0;$$

$$\left(\frac{\partial(x^\lambda)}{\partial(x^{\mu'})}\right)_{\mathfrak{x}} = \delta^\lambda_{\mu'}$$

bestehen. Falls s der affine Parameter einer Bahnkurve ist und $s = u^0/u^1$ gesetzt wird, so ergibt sich, dass eine Bahnkurve durch $\mathfrak{x}^\lambda$ in der Form $C_0^\lambda u^0 + C_1^\lambda u^1$ darstellbar ist, wo C_0^λ und C_1^λ Konstanten sind. Verf. zeigt, dass zwei Normalkoordinatensysteme derselben Punktes, ebenso wie im Falle affinzusammenhängender Räume, durch eine lineare Transformation mit konstanten Koeffizienten zusammenhängen. Falls $\Pi^\lambda_{\mu\nu}$ symmetrisch sind, können von den normalen Koordinaten ausgehend Normaltensoren und Erweiterungen von Tensoren wie in affinzusammenhängenden Mannigfaltigkeiten erklärt werden. *O. Varga* (Debrecen).

Nasu, Yasuo. Non-Euclidean geometry in Finsler spaces. Kumamoto J. Sci. Ser. A. **1**, no. 1, 8–12 (1952).

Verf. zeigt, dass die Übertragungsparameter einer projektivzusammenhängenden Linienelementmannigfaltigkeit, deren Holonomiegruppe eine nicht ausgeartete Hyperfläche Q_{n-1} zweiter Ordnung fest lässt, diejenigen eines Finslerschen Raumes sind. Ist das bewegliche Bezugssystem durch die Punkte $[R_0 R_j]$, $j = 1, 2, \cdots, n$, bestimmt, so lässt sich die Übertragung, bei geeigneter Wahl der Hyperfläche in der Form

$$dR_0 = R_i dx^i$$
$$dR_j = g_{jk} dx^k R_0 + (\Gamma^i_{jk} dx^k + C^i_{jk} dp^k) R_i$$

darstellen. Die Γ^i_{jk} und C^i_{jk} sind dabei die zum Fundamentaltensor g_{jk} gehörigen Cartanschen Übertragungsparameter. Die Metrik ist nach S. Sasaki, die einer nichteuklidischen Geometrie, deren absolutes Gebilde die ausgezeichnete Hyperfläche Q_{n-1} ist. *O. Varga* (Debrecen).

Takano, Kazuo. Homothetic transformations in Finsler spaces. Rep. Univ. Electro-Commun. **1952**, no. 4, 61–70 (1952). (Japanese summary)

Eine Punkttransformation des Finslerschen Raumes heisst homothetisch falls das Verhältnis entsprechender Bogenlängen konstant ist. Unter Benützung der, von E. T. Davies [siehe Ann. Mat. Pura Appl. (4) **18**, 261–274 (1939);

diese Rev. 1, 176] auf Finslersche Räume ausgedehnten Lieschen Ableitung, kann die Bedingung, dass die infinitesimale Punkttransformation (1) $\bar{x}^\lambda = x^\lambda + \xi^\lambda(x)\delta t$ homothetisch ist, durch $Xg_{\mu\nu} = 2cg_{\mu\nu}$ ausgedrückt werden. Hier ist X die zu (1) gehörige Liesche Ableitung. Hieraus ergibt sich, dass ein Finslerscher Raum, der eine infinitesimale homothetische Transformation gestattet, eine einparametrige, homothetische Transformationsgruppe besitzt. Ferner folgt als notwendige und hinreichende Bedingung für das Vorhanden sein einer einparametrigen, homothetischen Transformationsgruppe die Existenz eines Koordinatensystems, indem der metrische Fundamentaltensor $g_{\mu\nu}$ die Gestalt $g_{\mu\nu} = e^{2cx^1}F_{\mu\nu}(x^2, \cdots, x^n, \dot{x}^1 \cdots \dot{x}^n)$ besitzt. Eine zweite notwendige und hinreichende Bedingung ist die, dass die Komponenten des Fundamentaltensors, in einem geeigneten Koordinatensystem in den x^λ homogen von $2(c-1)$-ter Dimension sind, falls c die die homothetische Transformationen charakterisierende Konstante ist. Nach Verf. wird eine infinitesimale Transformation als konform bezw. affin bezeichnet falls $Xg_{\mu\nu} = \phi g_{\mu\nu}$ bezw. $X\Gamma^*{}_\mu{}^\lambda{}_\nu = 0$ gilt. Es ergibt sich dann, dass eine Transformation dann und nur dann homothetisch ist, falls sie zugleich konform und affin ist.

Ausser diesen Sätzen leitet Verf. noch einige Sätze her, die sich auf die Liesche Ableitung des Torsions bezw. Krümmungstensors des Raumes beziehen. *O. Varga.*

Nobuhara, Tetsuro, and Nagai, Tamao. On the special Finsler space of three dimensions. Tensor (N.S.) **2**, 175–180 (1952).

Für eine zweidimensionale Finslersche Mannigfaltigkeit lässt der, mit dem Einheitsvektor des Stützelementes l^i überschobene Krümmungstensor $R_0{}^i{}_{jk}$ bezw. der Torsionstensor eine recht einfache Beindarstellung zu, falls als Vektoren des Zweibeins der Vektor l^i und der zu ihm transversale Vektor verwendet wird. Die bei dieser Beindarstellung auftretenden Skalare sind der Berwaldsche Haupt- bezw. Krümmungsskalar. Im Falle eines dreidimensionalen Finslerschen Raumes kann man eine entsprechende Zerlegung nur bei Zugrundelegung einer Kurve durchführen. Als Dreibein kann der Einheitstangentenvektor, der Einheitsvektor in Richtung des Eulerschen Vektors und der zu diesen orthogonale Vektor verwendet werden. Bezeichnet man diese Vektoren der Reihe nach mit l^i, ρ^i und σ^i, dann zeigen die Verf., dass die Relationen

$$(1) \qquad R_0{}^i{}_{jk}\sigma^k = R_0{}^i{}_{jk}\sigma_i = A_{ijk}\sigma^k = 0$$

die notwendigen und hinreichenden Bedingungen dafür sind, dass sich $R_0{}^i{}_{jk}$ und $A_j{}^i{}_k$ formal ebenso durch das Dreibein darstellen lassen, wie dies im zweidimensionalen Falle mit Hilfe des Zweibeins möglich war. Die Beindarstellungen sind dabei

$$R_0{}^i{}_{jk} = \Re\rho^i(l_j\rho_k - l_k\rho_j), \quad A_{ijk} = \Im\rho_i\rho_j\rho_k.$$

Es wird nachgewiesen, dass sich die Bedingungen (1) auf ein System von neun Gleichungen reduzieren lassen, von denen höchtens drei linear unabhängig sind. Die Diskussion der Gleichungen ergibt folgendes: Falls drei linear unabhängige Gleichungen existieren, dann muss die betrachtete Kurve eine Extremale sein. Im Falle das System zwei linearunabhängige Gleichungen besitzt, muss für den Finslerschen Raum die Relation

$$R_{0ijk} = R_{0ijl}l^l l_k - R_{0ikl}l^l l_j$$

bestehen. Sie bedeutet, dass l^i ein invarianter Vektor der Holonomiegruppe des Raumes ist. Die betrachtete Kurve muss die Lösung einer gewissen Differenzialgleichung zweiter Ordnung sein. Wenn schliesslich sämtliche Gleichungen Vielfache einer einzigen Gleichung sind, dann existiert im Finslerschen Raum ein absoluter Parallelismus der Linienelemente l^i. Die möglichen Kurven sind wieder Lösungen von Differenzialgleichungen zweiter Ordnung vom selben Typus wie im vorangehenden Fall.

O. Varga (Debrecen).

***Galvani, O. La réalisation des espaces de Finsler.** Comptes Rendus du Congrès des Sociétés Savantes de Paris et des Départements tenu à Grenoble en 1952, Section des Sciences, pp. 57–60. Gauthier-Villars, Paris, 1952.

The author's general analysis of the realizations of Finsler spaces is applied here to a special case in which a Finsler F_3 is realized in a Euclidean E_3. The varieties in E_3 which correspond to elements of F_2 are the sets $S = (M, \Delta, P)$ where M is a point, Δ is a line through M, and P a plane through Δ. A number of results concerning these are stated without proof. *C. B. Allendoerfer* (Seattle, Wash.).

Barthel, Woldemar. Über eine Parallelverschiebung mit Längeninvarianz in lokal-Minkowskischen Räumen. I, II. Arch. Math. **4**, 346–354, 355–365 (1953).

Consider a Finsler space given by an integrand $L(x, \dot{x})$, where $L(x, X)$ is of class C^3, $L(x, X) > 0$ for $X \neq 0$, $L(x, \lambda X) = |\lambda| L(x, X)$, and $[\partial^2 L^2(x, X)/\partial X_i \partial X_k]\xi_i\xi_k > 0$ for $\xi \neq 0$. Let $F^{(p)}(x, Y) = F^{(p)}(x, Y_1, \cdots, Y_p)$ be the volume of the parallelepiped in the local Minkowskian geometry at x spanned by the vectors $Y_1, \cdots, Y_p$ with origin x. The properties of $F^{(p)}$ as a function of $Y_1, \cdots, Y_p$ are, of course, the same as in Minkowskian geometry. The function $F^{(n-1)}(x, Y)$ may also be written as a function $f^{(n-1)}(x, p)$ of a covariant vector density $p_k = \|\delta_k{}^i, Y_1{}^i, \cdots, Y^i{}_{n-1}\|$.

Following the reviewer the first curvature of a curve $x(t)$ is defined by means of the Minkowskian sine function. It is proved to equal $F^{(2)}(x, \dot{x}, D\dot{x}/dt)L^{-3}(x, \dot{x})$ where $D\dot{x}^i/dt = d\dot{x}^i/dt + [{}^i_{rs}]\dot{x}^r\dot{x}^s$, and the Christoffel symbols $[{}^i_{rs}]$ are defined in the usual way from the g_{ik} occurring in $L^2(x, X) = g_{ik}(x, X)X^iX^k$. The distribution of the normal curvatures of the curves on a hypersurface at a point is investigated. There appears a difference between the Minkowskian and the general Finsler case. [Reviewer's remark: This is traceable to the non-admissibility of normal coordinates in general Finsler spaces.] The totally geodesic hypersurfaces are characterized.

The second part introduces first another fundamental form $G^{ik}(x, p) = [\partial^2 f^{(n-1)}(x, p)^2/\partial p_i \partial p_k](2F^{(n)}(k)^2)^{-1}$. The question whether $G^{ik}(x, p)\xi_i\xi_k > 0$ for $\xi \neq 0$ is left undecided, but all further derivations assume this inequality. The analogue to Weingarten's equation is obtained in terms of the invariant differential

$$\mathfrak{D}p^i = dp^i + p^k\Gamma^i{}_{kh}(x, p)dx^h,$$

where

$$\Gamma^i{}_{kh}(x, p) = \{{}^i_{kh}\} + \{{}^r_{kh}\}p_r C_k{}^{ri},$$

$\{{}^i_{kh}\}$ are the Christoffel symbols belonging to G^{ik}, and $C_k{}^{ri}$ is derived from

$$C^{ikh}(x, p) = [\partial^3 f^{(n-1)}(x, p)^2/\partial p_i \partial p_k \partial p_h]/(2F^{(n)}(x)^2)^{-1}.$$

In an analogous way an invariant differential DX^i is obtained from the first fundamental form. Then parallel displacements are defined by $DX^i = 0$ and $\mathfrak{D}p^i = 0$; the former leaves length invariant, the latter preserves the $(n-1)$-dimensional volume of a vector density. Next, invariant

differentiations of both types are defined for tensors; uniqueness is obtained by imposing Cartan's postulates. The relation to Cartan's parallelism is discussed and it is shown that both parallel displacements coincide in the Riemannian case with the usual concept of Levi-Cività. *H. Busemann.*

Busemann, Herbert. Metrics on the torus without conjugate points. Bol. Soc. Mat. Mexicana **10**, 12–29 (1953). (Spanish)

M. Morse and G. Hedlund [Trans. Amer. Math. Soc. **51**, 362–386 (1942); these Rev. **3**, 309] and E. Hopf [Proc. Nac. Acad. Sci. U. S. A. **34**, 47–51 (1948); these Rev. **9**, 378] proved that if a riemannian metric on the torus is without conjugate points, it is euclidean. The author analyzes the same problem for the case of Finsler metrics. The principal results are: (a) The geodesics of a Finsler metric without conjugate points on the torus are not necessarily straight lines; (b) there is a great variety of different metrics without conjugate points with the same system of geodesics; (c) the systems of curves on the torus which can be geodesics of a Finsler metric without conjugate points are characterized by a given set of conditions. *L. A. Santaló.*

***Nalli, Pia. Lezioni di calcolo differenziale assoluto.** Tipografia Zuccarello & Izzi, Catania, 1952. 189 pp.

This is a very elementary course on tensor calculus as it was about 35 years ago and it covers the ground of what was interesting at that time. There are no indications to more modern conceptions or problems and there is not a single reference to literature. *J. A. Schouten* (Epe).

NUMERICAL AND GRAPHICAL METHODS

***Mineur, Henri. Techniques de calcul numérique à l'usage des mathématiciens, astronomes, physiciens et ingénieurs. Suivi de quatre notes par: Mme. Henri Berthod-Zaborowski, Jean Bouzitat, et Marcel Mayot.** Librairie Polytechnique Ch. Béranger, Paris et Liège, 1952. xxii+605 pp. 7300 francs.

There are few general treatises, and until now none in French, devoted to methods of interpolation and procedures for numerical calculation. The aim of the present work is to make available to others the practical experience of the author, who, as an astronomer, was led to study methods of calculation and as director of a mathematical research laboratory for national defense had further occasion to solve many numerical problems. The text is addressed not only to mathematicians but also to astronomers, physicists, engineers and others who encounter numerical calculations.

The author's plan of presentation is to divide each chapter or subdivision of a chapter into three parts, first theory, second technique, and third numerical applications. The theoretical part provides the derivation of the necessary mathematical formulas, the technique indicates the program by which the problem is set up as well as the organized routine for performing the calculation, and the numerical applications illustrate by specific examples the actual operation of the process.

A large part of the book is devoted to the presentation of interpolation and its applications. The author has chosen to base this presentation on symbolic methods, as was done in Boole's classic text. The first chapter develops the theory of operators as applied to interpolation, with application to the rapid summation of alternating series. The second chapter gives the formulas for interpolation of a function with values given at equal intervals, the third chapter takes up Lagrange's formula, and the fourth deals with numerical differentiation. In the fifth and sixth chapters the author explains methods for solving algebraic equations. Here he takes up Horner's method, solution of the cubic by radicals and by trigonometry, solution of the quartic by radicals, Sturm's sequences, and Graeffe's method.

The seventh chapter treats inverse interpolation, the eighth introduces orthogonal polynomials with a weight function, the ninth takes up numerical integration with equal intervals, and the tenth gives formulas for the integral of an integral. Chapter eleven introduces Bernoulli numbers, sums of powers of integers, and Eulerian integrals. In chapter twelve the author returns to numerical integration, taking up the Newton-Cotes formulas, the Euler-Maclaurin formula, the case of unequal intervals, Čebyšev's formula, and Gauss's method of numerical integration. Chapter thirteen gives the most complete treatment that the reviewer has so far seen of diverse methods for numerical integration when the integrand has various types of singularities. This treatment makes considerable use of the orthogonal polynomials given in chapter eight.

The numerical solution of differential equations is presented in chapter fourteen. The author limits his explanation to four methods, the method of Taylor's series, Adams' method, Moulton's method, and the Runge-Kutta method. The construction of tables of a function is the theme of chapter fifteen, methods for smoothing functional values occupy chapter sixteen, and interpolation for the case of two variables appears in chapter seventeen.

The book concludes with four articles by other authors on indeterminate forms, Čebyšev's method of integration, evaluation of certain singular integrals and extension of Gauss's method of integration.

The explanations are given in full and careful detail and with the clarity of exposition which one expects to find in a French text. The subject matter is strictly classical. Little note is taken of some recent trends in computational methods, trends first inspired by the advent of the electric desk calculator, and now accelerated by the astounding possibilities of large scale electronic computing machinery. Nevertheless the book is a valuable and welcome addition to the literature of numerical methods. *W. E. Milne.*

***Tables of 10^x (antilogarithms to the base 10).** National Bureau of Standards Applied Mathematics Series No. 27. U. S. Government Printing Office, Washington, D. C., 1953. viii+543 pp. $3.50.

This volume contains two tables of antilogarithms. The first table gives 10-decimal values for $x=0(0.00001)1$. It is based on the outstanding "antilogarithmic canon" of James Dodson published in London in 1742. The second table is a 15-decimal radix table, giving 10^x for $x=n\times 10^{-p}$, $n=1(1)999$, $p=3, 6, 9, 12, 15$.

The main impetus for the production of this volume was the inaccessibility and known errors in Dodson's Canon. This Canon was checked throughout by differencing and by the formation of sums, as a result of these checks about 250 major errors were found and removed; it is surprising that the list of errors is not given.

The checking processes applied in producing this useful volume were not capable of detecting errors in the tenth

decimal. Evidence that this last digit is not always rounded correctly is found by intercomparison of tables I and II. It was clearly not worth while to undertake the large amount of work needed for a slight improvement possible in the tenth decimal, but surprising that the general retention of Dodson's last figure is not mentioned specifically.

A brief introduction accompanies the tables.

J. C. P. Miller (Cambridge, England).

Bleick, Willard Evan. Tables of associated sine and cosine integral functions and of related complex-valued functions. United States Naval Postgraduate School, Monterey, Calif., Tech. Rep. No. **10**, ii+103 pp. (1953).

This volume contains the following three tables. I) Real and imaginary parts of $\mathrm{Ci}(x+jy)=\int_{\infty}^{x+jy}z^{-1}\sin z\,dz$; II) real and imaginary parts of $\mathrm{Si}(x+jy)=\int_{0+j0}^{x+jy}z^{-1}\sin z\,dz$, $z=x+jy$, and the paths of integration parallel to the real axes, i.e., $dz=dx$; III) the functions

$$\mathrm{Cia}(x,y)=\int_{\infty}^{x}(t^2+y^2)^{-1}t\cos t\,dt=\mathrm{Re}\,\mathrm{Ci}(x+jy)\cosh y$$
$$-[\mathrm{Im}\,\mathrm{Si}(x+jy)+\tfrac{1}{2}(\mathrm{Ei}(y)-\mathrm{Ei}(-y))]\sinh y,$$

where $\mathrm{Ei}(y)=\int_{-\infty}^{y}e^{-t}t^{-1}dt$,

$$\mathrm{Sia}(x,y)=\int_{0}^{x}(t^2+y^2)^{-1}t\sin t\,dt=\mathrm{Re}\,\mathrm{Si}(x+jy)\cosh y$$
$$+(\mathrm{Im}\,\mathrm{Ci}(x+jy)-\pi/2)\sinh y.$$

These functions are tabulated to 12D for $x=0(.1)3.1$, and $y=0(.1)3.1$, but the accuracy is guaranteed to not more than 10D. It is stated that: "It is anticipated it will be possible to tabulate these functions in a future report over a more extended range."

The author's definition of $\mathrm{Si}(z)$ differs from the normal one, which we shall call $\mathrm{Si}(z)$. Then the author's function is equal to $\mathrm{Si}(z)+\frac{1}{2}i[\mathrm{Ei}(y)-\mathrm{Ei}(-y)]$. It is claimed that the functions $\mathrm{Sia}(x,y)$, $\mathrm{Cia}(x,y)$ were first introduced by J. G. Chaney in his studies of rhombic antennas, 1953.

R. C. Archibald (Providence, R. I.).

Mashiko, M. Tables of generalized exponential sine- and cosine-integrals $\mathrm{Ei}(x+iy)$, $\mathrm{Si}(x+iy)$, $\mathrm{Ci}(x+iy)$. Numerical Computation Bureau, Tokyo, Report no. **7**, 43 pp. (1953).

$$-\mathrm{Ei}(-z)=I(z)=\int_{z}^{\infty}e^{-t}t^{-1}dt=\mathrm{C}\alpha(\xi)-i\mathrm{S}\alpha(\xi)$$
$$=e^{-z}z^{-1}\mathrm{A}\alpha(\eta)e^{i\Phi\alpha(\eta)},$$

where $z=\xi e^{i\alpha}=\eta^{-1}e^{i\alpha}$. Tables are given for: I) $\mathrm{C}\alpha(\xi)+\ln\xi$, and $\mathrm{S}\alpha(\xi)$ to 6D, with δ^2, $\xi=0(.05)5$, $\alpha=0°(2°)60°(1°)90°$; II) $\mathrm{A}\alpha(\eta)$ to 6D, $\Phi\alpha(\eta)$ to 5D, each with δ^2, for the previous α range, and $\eta=0(.01).2$; in indicating ranges the author makes four slips in inequality signs; III) Everett second-order interpolation coefficients, 4D, at interval .01. If accurate these are new tables of value. *R. C. Archibald.*

Glowatzki, Ernst. Tafel der Jacobischen elliptischen Funktion $\varphi=am\left(\frac{m}{n}K\right)$. Abh. Bayer. Akad. Wiss. Math.-Nat. Kl. (N.F.) no. **61**, 27 pp. (1953).

Table to three decimals of a degree of Jacobi's

$$\phi=\mathrm{am}[m\cdot K/n;\sin\Theta]$$

for $n=2(1)12$, $\Theta=0°(1°)90°$ to enable Jacobi's elliptic functions to be calculated from trigonometric tables. [The author seems to be unaware of the existence of the reviewer's "Die elliptischen Funktionen von Jacobi" [Springer, Berlin, 1931] and "Jacobian elliptic function tables" [Dover, New York, 1950; these Rev. **13**, 987] which give results directly.]

L. M. Milne-Thomson (Greenwich).

Bates, D. R., Ledsham, Kathleen, and Stewart, A. L. Wave functions of the hydrogen molecular ion. Philos. Trans. Roy. Soc. London. Ser. A. **246**, 215–240 (1953).

The wave functions in question satisfy the differential equations

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dM}{d\mu}\right]+\left[-A+p^2\mu^2-\frac{m^2}{1-\mu^2}\right]M=0,$$

$$\frac{d}{d\lambda}\left[(\lambda^2-1)\frac{d\Lambda}{d\lambda}\right]+\left[A+2R\lambda-p^2\lambda^2-\frac{m^2}{\lambda^2-1}\right]\Lambda=0,$$

where m is a given integer, R is the distance between two nuclei in atomic units, E is the electron energy in Rydbergs, $p^2=-\frac{1}{4}R^2E$, and A is a separation constant. Solutions of these equations are obtained in the form

$$M=\sum f_r P^m_{m+r}(\mu),\quad \Lambda=(\lambda^2-1)^{\frac{1}{2}m}(\lambda+1)^{\sigma}e^{-p\lambda}\sum g_r\left(\frac{\lambda-1}{\lambda+1}\right)^r,$$

where $\sigma=R/p-m-1$.

The μ-equation is the differential equation of spheroidal wave functions. For a given m, p, one obtains characteristic values A_l which were tabulated by Stratton, Morse, Chu, and Hutner [Elliptic cylinder and spheroidal wave functions, Wiley, New York, 1941; these Rev. **4**, 281]. Once A is known, the λ-equation determines characteristic values R_n. Computations were carried out for a number of combinations n, l, m, p, pertaining to ten "states". The corresponding values of R were obtained and by inverse interpolation the basic parameters, p, σ, $-A'=p^2-A$, and $-E$, were expressed as functions of R. Table 1 gives 5D values of these parameters for $R=0(.2)5(.5)10$ for the ten states $1s\sigma_g$, $2s\sigma_g$, $3s\sigma_g$, $2p\sigma_u$, $3p\sigma_u$, $4p\sigma_u$, $3d\sigma_g$, $4f\sigma_u$, $2p\pi_u$, $3d\pi_g$. Table 2 gives the coefficients f_r of the μ-expression, and Table 3 the coefficients g_r of the λ-expression, for these ten states.

A. Erdélyi (Pasadena, Calif.).

Ekelöf, Stig. Theory of electromagnetically delayed telephone relays. A study of telephone relays. II. Ericsson Technics **9**, 141–224 (1953).

[For part I see Ericsson Technics **9**, 51–82 (1953).] This paper contains graphs (on p. 161) and 4D numerical tables (at the end of the paper) of the functions $\cosh x$, $x^{-1}\sinh x$, $3x^{-2}\cosh x-3x^{-3}\sinh x$, $2x^{-2}(\cosh x-1)$, and $12x^{-3}\sinh x-24x^{-4}(\cosh x-1)$ in the interval $0\leqq x\leqq 1.5$.

A. Erdélyi (Pasadena, Calif.).

***Beers, Yardley. Introduction to the theory of error.** Addison-Wesley Publishing Co., Inc., Cambridge, Mass. vi+65 pp. $1.25.

This compact and practical little book was written by a physicist primarily for physicists, although it has value for any one having to make critical analyses of instrumental observations. The book starts out by defining and illustrating the technical terms employed. It then proceeds to a classification and discussion of errors, treating experimental errors in measured quantities, deviations, standard deviation, and average deviation, with examples. The normal error curve is first arrived at heuristically rather than by mathematical reasoning, but later is derived from the basic assumptions about errors. The standard deviation and average deviation are found for the normal curve. The next

chapter takes up propagation of error and the errors of functions of several variables, with applications to the selection of instruments or techniques. There follows a chapter on the adjustment of data, taking up weighted averages, least-square fit to a straight line, and applications thereof. The next chapter deals with the statistical errors of nuclear physics and introduces the binomial distribution. The book concludes with a couple of examples analysed in some detail, the torsion pendulum experiment and the calibration of a Jolly balance. *W. E. Milne.*

Riguet, Jacques. Sur les rapports entre les concepts de machine de multipole et de structure algébrique. C. R. Acad. Sci. Paris **237**, 425–427 (1953).

Proposition: The concept of "Ashby machine" [Ashby, Design for a brain, Wiley, New York, 1952] is equivalent to that of algebraic structure (abstract algebra), as defined by the reviewer [Proc. Cambridge Philos. Soc. **31**, 433–454 (1935)]. *G. Birkhoff* (Cambridge, Mass.).

Miller, K. S., and Murray, F. J. A mathematical basis for an error analysis of differential analyzers. J. Math. Physics **32**, 136–163 (1953).

This paper attacks the problem of just what is the relation between the solution to a system of ordinary differential equations as obtained from the typical analog type of differential analyser, and the "true" solution. The machine in the model used is assumed to make only three types of errors: α errors due to not putting in exactly the right coefficients, β errors which are sudden jumps from one solution to an adjacent solution and are attributed to "noise" in the machine, and λ errors which are changes in the order of the equations being solved due to imperfect integrators. The discussion of the effects of these errors is quite formal and mathematical. *R. Hamming.*

Juncosa, M. L. Random number generation on the BRL high-speed computing machines. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. No. 855, 25 pp. (1953).

This report describes procedures used by the Ballistic Research Laboratories to produce random sequences for Monte Carlo methods. Those processes which involve physical phenomena such as cosmic radiation or the shot effect have been rejected in favor of purely mathematical routines requiring a minimum of storage space for orders and working space. It is pointed out that the "pseudo-random" numbers produced by a definite algorithm can be re-run and tested for randomness if desired.

The methods used on the three large-scale computers differ in detail only and depend upon the theory of the binomial congruence. One starts with an initial integer r_0 which is prime to the base b of the number system used in the machine. Then r_{i+1} is produced via the congruence

$$r_{i+1} \equiv kr_i \pmod{b^\beta},$$

where b^β is the "word size" of the machine, i.e., β is the number of digits in a typical machine number, and k is chosen so as to make the period of the r's a maximum. Then the numbers $x_i = r_i b^{-\beta}$ lie between 0 and 1 and are the pseudo-random numbers of the process. For the ORDVAC, $b=2$, $\beta=39$, and $k=5^{13}$. For the EDVAC, $b=2$, $\beta=43$, $k=5^{17}$, for the ENIAC, $b=10$, $\beta=10$, $k=[1+10(4\lambda+1)]^n$, where λ is not divisible by 5. These choices give periods of length 2^{37}, 2^{41}, and $5\cdot 10^8$ respectively. In the case of the ORDVAC the program requires 4 (single address) orders and 4 storage positions. The EDVAC requires only one order and 4 storage positions. The multipliers k are chosen large so as to avoid correlations in size between successive x's. Various statistical tests were run on some of these sequences. These have to do with uniform distribution of frequencies and the ability of sums to produce normal distribution. All tests seem to give satisfactory results.

The more popular "middle square" method of generating random numbers was rejected because of the possibility of the degeneration of the sequence into a null sequence. There is an additional advantage of the methods used over the middle square method which the author could have indicated, namely, it is easy for the coder to anticipate the exact value of x_n given x_0 and n. Thus at the end of a program involving a definite number n of steps, no matter how large, the machine may be asked to print out its value of x_n as a check on its accuracy.

The method used is subject to a disadvantage when a source of random integers is needed. Indeed, all the r_i are odd and the low-ordered digits are subject to short periods. This can be overcome by adopting a modulus like $b^\beta \pm 1$ as suggested by the reviewer in 1949 [cf. these Rev. **13**, 495]. *D. H. Lehmer* (Berkeley, Calif.).

Piloty, Robert. Ein neues Multiplikationsverfahren für Dualzahlen. Z. Angew. Math. Mech. **33**, 429–430 (1953).

Youse, B. K. Abridged series for numerical evaluation. Amer. Math. Monthly **61**, 187 (1954).

Collatz, Lothar. Einige Anwendungen funktionalanalytischer Methoden in der praktischen Analysis. Z. Angew. Math. Physik **4**, 327–357 (1953).

The author reviews the work of many others and himself on the use of functional analysis to prove the existence of unique solutions of various functional equations, and especially to give numerical bounds for the solutions. The equations considered include linear and nonlinear systems of simultaneous equations, initial-value and boundary-value problems for nonlinear ordinary and partial differential equations, and the linear Fredholm integral equation of the second kind.

Much of the treatment is unified in terms of an abstract normed additive group R. In a complete subset F of R there is a nonlinear operator T with the Lipschitz property $\|Tf_1 - Tf_2\| \leq K\|f_1 - f_2\|$, $K<1$. It is known that the equation $Tu=u$ has a unique solution u in F, that $\|u_n - u\| \to 0$, where $u_{n+1} = Tu_n$, and that $\|u - u_1\| \leq K(1-K)^{-1}\|u_1 - u_0\|$. The various applications are characterized by different choices of norms, and by coarser or finer estimates of K. The mean pth power norms ($p=1$, 2, and mostly ∞) are used in analyzing the iterative solution of simultaneous equations by "entire steps" or "single steps", and by the linearized form of Newton's method in which one inverts a Jacobian matrix only once. For function spaces the author uses norms of type $\|y\| = \sup_x (|y(x)|/W(x))$, where $W(x)$ is a carefully selected positive function.

The rest of the treatment is unified in terms of a partially ordered function space in which $u_1 \leq u_2$ means pointwise inequality. The nonlinear operator T is said to be of "monotone kind" if $Tv \leq Tw$ implies $v \leq w$. Then, if $Tv_1 \leq f \leq Tv_2$, the solution of the functional equation $Tu=f$ satisfies the relations $v_1 \leq u \leq v_2$. There are applications to linear systems of equations and to boundary-value problems.

There are numerical examples for most of the applications, and a bibliography of 26 titles. In this connection, the

reviewer suggests also Kantorovich [Uspehi Matem. Nauk (N.S.) 3, no. 6(28), 89–185 (1948) = Nat. Bur. Standards, Los Angeles, Calif., NBS Rep. 1509 (1952); these Rev. 10, 380; 14, 766]. *G. E. Forsythe* (Los Angeles, Calif.).

Schröder, Johann. Eine Bemerkung zur Konvergenz der Iterationsverfahren für lineare Gleichungssysteme. Arch. Math. 4, 322–326 (1953).

Let the finite linear system $Ax+r=0$ with nonvanishing determinant have all $a_{jj}<0$. Consider the hypotheses: (I) all $a_{jk}=\bar{a}_{kj}$; (II) $a_{jk}\geqq 0$ for $j\neq k$; (III) no matrix PAP is triangular in blocks, where P is a permutation matrix. Let C, B be the lower, upper triangles of A, and let $-D$ be the diagonal of A, so that $A=C-D+B$. Then solution of the system in "entire steps" follows the iteration $Dx^{k+1}=Cx^k+Bx^k+r$. The solution in "single steps" follows the iteration $Dx^{k+1}=Cx^{k+1}+Bx^k+r$. The author proves that, if the entire-step iteration converges, then either (I) or (II)·(III) implies that the single-step iteration converges. Moreover, if either (II)·(III) or (I)·(II) holds, and if both iterations converge, then (*): single-step iteration is not slower to converge than the entire-step method. Some further estimates are given.

Reviewer's note: In Theorem VI of Stein and Rosenberg [J. London Math. Soc. 23, 111–118 (1948); these Rev. 10, 485] it is apparently shown that (II) alone implies both the equiconvergence of the two iterations and a stronger form of (*). *G. E. Forsythe* (Los Angeles, Calif.).

Curtiss, J. H. "Monte Carlo" methods for the iteration of linear operators. J. Math. Physics 32, 209–232 (1954).

If the eigenvalues of the square matrix H lie inside the unit circle a sequence of vectors u_N $(N=0,1,\cdots)$ that satisfies the recursion relations $u_{N+1}=Hu_N+c$ tends to the vector u for which (1) $u=Hu+c$. On the basis of this well-known fact two slightly different sampling techniques for the numerical inversion of matrices have been proposed [G. E. Forsythe and R. A. Leibler, Math. Tables and Other Aids to Computation 4, 127–129 (1950); these Rev. 12, 361; W. Wasow, ibid. 6, 78–81 (1952); these Rev. 14, 1018]. In the first part of the present paper the author formulates and rearranges these procedures in a manner that helps to clarify their relationship. His formulation enables him, in the later sections, to give explicit expressions for the variances of these procedures which are more general than those obtained by the authors quoted. He then proposes, for a certain class of matrices H, a new sampling procedure, related to that of Forsythe and Leibler, for the solution of (1), in which the variance is made smaller by an "importance sampling" technique based on the residuals for the starting vector u_0. In an introductory section a similar error analysis is applied to statistical sampling methods for numerical quadrature.

W. Wasow (Los Angeles, Calif.).

Unger, Heinz. Zur Praxis der Biorthonormierung von Eigen- und Hauptvektoren. Z. Angew. Math. Mech. 33, 319–331 (1953). (English, French and Russian summaries)

Assume we know a certain eigenvalue λ of an arbitrary real matrix $\mathfrak{A}$. The problem is to compute complete biorthonormal bases for the invariant spaces of $\mathfrak{A}$ and $\mathfrak{A}'$ belonging to λ. This problem is important in various iterative and direct methods of computing the eigenvalues of $\mathfrak{A}$. In the general case of nonlinear divisors matters are complicated. The author's method is fully expounded for real λ. One first computes the eigenvectors by an abridged Gaussian elimination solution of the systems $(\mathfrak{A}-\lambda\mathfrak{E})\xi=0$ and $(\mathfrak{A}'-\lambda\mathfrak{E})\eta=0$. This suffices when the elementary divisors of $\mathfrak{A}$ are linear. For nonlinear divisors one then solves linear systems to generate two chains of principal vectors like $\xi^1, \xi^2, \cdots$, characterized by $(\mathfrak{A}-\lambda\mathfrak{E})\xi^\sigma=\xi^{\sigma-1}$. A process based again on the Gaussian algorithm will biorthogonalize two corresponding chains. Further chains are obtained similarly, after being "purified" of all components along vectors in previously obtained chains. There is an explanation on dealing with complex eigenvalues, limited to linear divisors.

There are several computational schemata. For numerical examples one is referred to an unpublished report of the Institut für praktische Mathematik. *G. E. Forsythe.*

Caprioli, Luigi. Sulla risoluzione dei sistemi di equazioni lineari con il metodo di Cimmino. Boll. Un. Mat. Ital. (3) 8, 260–265 (1953).

To solve a system of linear equations (*) $\sum_{k=1}^n a_{hk}x_k=b_h$ with nonvanishing determinant, G. Cimmino [Ricerca Sci. (2) 1, 326–333 (1938)] proposed an iterative method which always converges. In it, $x^{(\nu)}$ is defined to be any weighted average of the n reflections of $x^{(\nu-1)}$ in the n hyperplanes (*). By a special choice of the weights one can write Cimmino's method in the form

$$x_h^{(\nu)}=x_h^{(\nu-1)}+p\sum_{k=1}^n a_{hk}z_k^{(\nu-1)},$$

where

$$z_h^{(\nu-1)}=b_h-\sum_{i=1}^n a_{hi}y_i^{(\nu-1)}.$$

While Cimmino took $p=2/\sum_{h,i=1}^n a_{hi}^2$, the author gives sufficient conditions for the convergence for arbitrary fixed p, and in particular for $p=1$. While the convergence will ordinarily require many steps, the author feels that this modified Cimmino method is well adapted to taking full advantage of the speed of modern electronic computers.

G. E. Forsythe (Los Angeles, Calif.).

Blanch, Gertrude. On modified divided differences. I. Math. Tables and Other Aids to Computation 8, 1–11 (1954).

The nth modified divided difference is $n!w^n$ times the nth ordinary divided difference. Here w is fixed by convenience, the author recommending a value approximately equal to the mean of the absolute differences between consecutive abscissae. The computational advantage is that all differences can be recorded (as are simple differences when the abscissae are equally spaced) to the same number of decimals. Four theorems are proved and applied to the search for errors in tabular entries. *A. S. Householder.*

Bertiau, F. New numerical integration methods. Simon Stevin 29, 196–202 (1952). (Dutch)

The numerical evaluation of the integral $J=\int_{x_0}^{x_n} f(x)dx$ is carried out by a formula of the type

$$J\approx p_0f_0+\sum_{i=1}^n[p_i(f_i+f_{-i})+q_i(f_i'-f_{-i}')]$$

with suitable weights p_i and q_i, where $f_i=f(ih)$ and $f_i'=f'(ih)$. *H. Bückner* (Schenectady, N. Y.).

Bertova, E. I., Kuznecov, Ya. T., Natanson, I. P., and Caregradskii, H. A. On approximate computation of definite integrals by means of a multiplicative method of excluding singularities. Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 639–644 (1953). (Russian)

This paper extends Gauss's method of numerical integration to the case of functions having certain types of singularities in the interval of integration. The method developed here applies to the case where the integrand can be written as a product of the form $|x|^{\alpha}f(x)$ in which $f(x)$ is continuous and α lies in the interval $-1<\alpha<0$. As is well known, one may approximate the integral by a sum of the form $\sum_{i=1}^{n}A_i f(x_i)$ in which the A's are constants and the x's are zeros of a polynomial $w_n(x)$ of degree n, where the set of polynomials $w_0(x), w_1(x), \cdots, w_n(x)$ are orthogonal with weight function $|x|^{\alpha}$ for the given interval of integration. Such polynomials can be constructed by a three-term recurrence relation. The author obtains the orthogonal polynomials through the eighth degree and tabulates values of the A's and x's for $\alpha=-\frac{3}{4}$, $-\frac{2}{3}$, $-\frac{1}{2}$, $-\frac{1}{3}$ and $-\frac{1}{4}$.

W. E. Milne (Corvallis, Ore.).

Bukovics, E. Beiträge zur numerischen Integration. I. Der Fehler beim Blaess'schen Verfahren zur numerischen Integration gewöhnlicher Differentialgleichungen n-ter Ordnung. Monatsh. Math. 57, 217–245 (1953).

The author asserts that there are three methods available for the approximate solution of differential equations with assigned initial values, namely, the method of differences, the Runge-Kutta method, and the method of Blaess. For the first two there have been developed methods for estimating the magnitude of the error committed. For the method of Blaess, however, there is no satisfactory error estimate. In this paper the author develops a usable estimate for the current error in the case of a differential equation of arbitrary order. First of all he obtains a recursive relation between the errors of successive steps, and from this, by solving a system of simultaneous linear difference equations, he gets expressions for the cumulative error. In order to facilitate the numerical evaluation of this expression the author supplies tables of certain parameters occurring in the error estimates for the cases of second, third, and fourth orders. *W. E. Milne* (Corvallis, Ore.).

Bukovics, Erich. Beiträge zur numerischen Integration. II. Der Fehler beim Runge-Kutta-Verfahren zur numerischen Integration gewöhnlicher Differentialgleichungen n-ter Ordnung. Monatsh. Math. 57, 333–350 (1954).

In part I [reviewed above] formulas are given for making current estimates of the error in Blaess's method. The present paper does the same thing for the Runge-Kutta method, employing the direct formulas derived by Zurmühl. These error estimates are compared with the expression obtained by Bieberbach for the error committed in a single step. Then the errors of the Blaess and of the Runge-Kutta method are compared. It is shown by examples that a true comparison of the methods can be made only by carrying out a number of steps of the computations.

W. E. Milne (Corvallis, Ore.).

Zadiraka, K. V. Solution by the method of S. A. Čaplygin of linear differential equations of the 2d order with variable coefficients. Dopovidi Akad. Nauk Ukrain. RSR 1951, 163–170 (1951). (Ukrainian. Russian summary)

The equation considered is $y''+Q(x)y=0$ with $y(a)=y_0$, $y'(a)=y_0'$, where Q is continuous and non-vanishing on the closed interval $[a, b]$. If $Q<0$, $m^2\leqq -Q$, the author defines a recursion by

$$\psi_k=t_k''+Qt_k,\quad \varphi_k=Z_k''+QZ_k,$$
$$\eta_{k+1}''-m^2\eta_{k+1}-\psi_k=\xi_{k+1}''-m^2\xi_{k+1}-\varphi_k=0,$$
$$\eta_k(a)=\eta_k'(a)=\xi_k(a)=\xi_k'(a)=0,$$
$$t_k=t_{k-1}-\eta_k,\quad Z_k=Z_{k-1}+\xi_k.$$

Then if t_0 and Z_0 satisfy initial conditions together with the differential inequalities $\psi_0\gtrless 0$, $\varphi_0\leqq 0$, the two sequences t_k and Z_k approach y monotonically and uniformly from above and below, respectively. If $Q>0$, sequences are constructed for the interval $[a, x_\nu]$, where x_ν is the smallest zero of y.

A. S. Householder (Oak Ridge, Tenn.).

Zadiraka, K. V. Computation of the characteristic values and functions of a Sturm-Liouville boundary problem. Dopovidi Akad. Nauk Ukrain. RSR 1951, 171–176 (1951). (Ukrainian. Russian summary)

The method outlined in the previous paper is extended to yield sequences approaching the first characteristic function and value for $y''+Q(x, \lambda)y=0$, $y(a)=y(b)=0$.

A. S. Householder (Oak Ridge, Tenn.).

Čečik, V. A. On applicability of S. A. Čaplygin's method to approximate integration of nonlinear partial differential equations of the first order. Doklady Akad. Nauk SSSR (N.S.) 91, 741–744 (1953). (Russian)

This paper extends Čaplygin's theory of differential inequalities to the case of non-linear partial differential equations of the first order in the form (1) $q=f(x, y, z, p)$. It is shown that if $z(x, y)$ is a solution of (1), and $v(x, y)$ is a solution of $q=f(x, y, z, p)+g(x, y)$, where $g(x, y)>0$, both z and v satisfying the same Cauchy boundary condition, then (with some additional restrictions) one may conclude that $z(x, y)\leqq v(x, y)$ in a certain region of the (x, y)-plane. Then the author proceeds to establish a sequence of functions $v_n(x, y)$ satisfying the inequality $v_{n+1}(x, y)<v_n(x, y)$, and approaching the solution $z(x, y)$. *W. E. Milne.*

Hu, Hai-Chang. On the approximate solutions of boundary value problems at a point. Acad. Sinica Science Record 5, 59–68 (1952). (Chinese summary)

A method is proposed for obtaining approximate solutions of boundary value problems at a point based on an idea first introduced by Prager and Synge [Quart. Appl. Math. 5, 241–269 (1947); these Rev. 10, 81] and further developed by Synge, Diaz, Greenberg and the reviewer.

Given the boundary value problem (1) $D\phi=q$ in R, with ϕ prescribed on the boundary B of R, where D is a certain differential operator; the object of the present paper is to approximate a functional $fp(\phi)$ at a point p contained in R. This is done by the introduction of a Green's function g through the equation (2) $fp(\phi)=\int_R qg\,dv$, and observing that the error $\delta fp(\phi)$ of the approximation to $fp(\phi)$ is given by (3) $fp(\phi)=\int_R \delta q\,\delta g\,dv$, where $q=q-q'$, $q'=D\phi'$, $g=g-g'$. The prime indicates an approximate quantity.

The author uses the non-homogeneous harmonic and the non-homogeneous biharmonic problems to illustrate the application of his method in a general way. In the first problem the functional $fp(\phi)$ required is the function ϕ itself while in the latter the functional $fp(\phi)$ required is the bending moment tensor.

C. G. Maple (Ames, Iowa).

Walsh, J. L., and Young, David. On the accuracy of the numerical solution of the Dirichlet problem by finite differences. J. Research Nat. Bur. Standards **51**, 343–363 (1953).

This paper derives numerical bounds for the truncation error in the solution of the Dirichlet problem, caused by replacing Laplace's equation by the simplest approximating difference equation. The error bounds obtained involve quantities which do not depend upon knowledge of the differential solution inside the region, such as the mesh size, and the oscillation and modulus of continuity of the prescribed boundary distribution. Explicit results are given for rectangles, but it is indicated how the methods can be extended to other straight-sided plane regions. Novelties of method are the use of the difference analogs of harmonic measure and the Schwarz alternating process. For the analog of the more general Kellogg alternating process, see J. B. Diaz and R. C. Roberts [Quart. Appl. Math. **9**, 355–360 (1952); these Rev. **13**, 388].

M. A. Hyman (Washington, D. C.).

Young, David. Iterative methods for solving partial difference equations of elliptic type. Trans. Amer. Math. Soc. **76**, 92–111 (1954).

This paper considers linear systems

$$(1)\qquad \sum_{j=1}^{N} a_{ij}u_j+d_i=0,\quad N \text{ large},$$

where $A=\|a_{ij}\|$ includes matrices of a sort frequently occurring in the solution of elliptic partial differential equations by difference methods (in particular, $a_{ii}\neq 0$). Rewriting (1) as

$$(2)\qquad u_i=\sum_{j\neq i} b_{ij}u_j+c_i,$$

one can iterate to find u_i ($i=1, 2, \cdots, N$, successively). If u_k^{m+1} ($k<i$, superscript is number of iteration cycle) are used to compute u_i^{m+1} (Gauss-Seidel method), convergence is asymptotically twice as rapid as when u_k^m are used. Also, one may "over-relax":

$$(3)\qquad u_i^{m+1}=\omega\Big[\sum_{j\neq i} b_{ij}u_j^m+c_i\Big]-(\omega-1)u_i^m,\quad 1<\omega<2,$$

a process proposed by Richardson [Philos. Trans. Roy. Soc. London. Ser. A. **210**, 307–357 (1910)] who suggested changing ω from time to time to speed up convergence. In the present paper over-relaxation (with fixed ω) is combined with immediate introduction of newly-computed u's, à la Gauss-Seidel. Various theorems on convergence are proved; in particular, it is shown that there exists an ordering of the equations and an optimum value ω_0 such that in general (3) converges much more rapidly than the Gauss-Seidel method ($\omega=1$). Means are suggested for estimating ω_0; the sensitivity of the rate of convergence to the choice of ω is studied. The paper concludes with a theoretical comparison of Gauss-Seidel and the method proposed, "successive over-relaxation", for solving Dirichlet's difference problem over a square using a high-speed computing machine. On page 109, factors of n^{-1}, n^{-1}, n^1 have been omitted from the second, third and fourth formulas, respectively.

M. Hyman (Washington, D. C.)

Young, David, and Warlick, Charles H. On the use of Richardson's method for the numerical solution of Laplace's equation on the ORDVAC. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Memo. Rep. No. **707**, 32 pp. (1953).

Richardson [Philos. Trans. Roy. Soc. London. Ser. A. **210**, 307–357 (1910)] proposed to solve the linear system $\sum_{j=1}^{N} a_{ij}u_j+d_i=0$, $i=1, 2, \cdots, N$, by the iteration scheme

$$u_i^{(n+1)}=u_i^{(n)}+\beta_n\Big[\sum_{j=1}^{N} a_{ij}u_j^{(n)}+d_i\Big];$$

the β_n are constants, varying from one iteration cycle to the next (denoted by superscripts); however, $\beta_{n+m}=\beta_n$, m a fixed constant. Using ORDVAC, Richardson's method was applied to solving the usual difference analog of the Dirichlet problem over a square, with $19^2=381$ interior points. For optimal m, β_n's, convergence to within prescribed accuracy was considerably faster than with Gauss-Seidel method, not so rapid as when using less general "successive-overrelaxation" [see the paper reviewed above]. Careful attention must be paid to growth of round-off errors when using Richardson's method.

M. A. Hyman.

Southwell, R. V., and Vaisey, Gillian. On some eigenvalue problems of exceptional difficulty, exemplified by a case of elastic instability. Quart. J. Mech. Appl. Math. **6**, 453–480 (1953).

The determination of the gravest critical force and mode of buckling of a deep rectangular cantilever, with edges clamped and stressed by uniform thrust applied to one of its longer sides, is used as a test example of difficult eigenvalue problems in which (a) the gravest mode has nodal lines, (b) the eigen-values may have either sign, (c) two or more eigen-values may be nearly equal in modulus. Other practical difficulties arise from the necessary use of a large number of mesh points in a relaxational solution of the partial differential equation, given here by $D_1w=\lambda D_2w$, where D_1 (fourth order) and D_2 (second order) are linear differential operators.

There is a discussion of the efficiency for such problems of (i) Rayleigh's principle, for determining the required λ from a guessed w, (ii) "intensification", for finding the "smallest" eigen-solution as the limit of the sequence defined by $D_1w_n=D_2w_{n-1}$, (iii) "partial transfer", using the equation $(D_1-p)w_n=D_2w_{n-1}$ to force convergence to the smallest positive eigen-value. Other methods include the elimination of unwanted components of a guessed mode, and the use of various modifications and extensions of Rayleigh's principle.

Difficulties (a) and (b) are surmounted by fixing the value of w at one well-chosen mesh point, taking a sequence of positive λ's, and relaxing so that $(D_1-\lambda D_2)w=0$ at all except the chosen point. The residual at this point is zero only if λ is an eigen-value, obtainable by inverse interpolation. Extensions of this process, using Rayleigh's theorems on the effect of constraints, give information about the position of nodal lines and the possible equality of positive eigen-values. The test example is shown to have two nearly equal positive roots λ_1 and λ_2 so that the mode of λ_1 is not determinable with great accuracy. Information about w_1, w_2 and w_{-1} (corresponding to the smallest negative root) is obtained from two "intensifications", and upper and lower limits on λ_1 are derived from the results of a separate paper [Southwell, same Quart. **6**, 257–272 (1953); these Rev. **15**, 257].

L. Fox (Teddington).

Miller, S. C., Jr., and Good, R. H., Jr. A WKB-type approximation to the Schrödinger equation. Physical Rev. (2) **91**, 174–179 (1953).

The connection between the solutions to the one-dimensional wave equation $d^2\psi/dx^2+Q(x,\lambda)\psi=0$, and the fundamental functions being solutions to the simplest cases $Q=1$, $Q=x$, $Q=\lambda-x^2$, etc., has been discussed by Löwdin [Quarterly progress report of solid-state and molecular theory group, Mass. Inst. Tech., January 15, 1952, p. 12]. The introduction of the transformation $x=x(t)$, $\psi(x)=(dx/dt)^{\frac{1}{2}}\phi(t)$ leads to another wave equation of the same form

$$\frac{d^2\phi}{dt^2}+R(t,\lambda)\phi\equiv\frac{d^2\phi}{dt^2}+\left[Q\left(\frac{dx}{dt}\right)^2+\tfrac{1}{2}\{x,t\}\right]\phi=0,$$

where $\{x,t\}$ is Schwarz' differential invariant [see, e.g., Forsyth, Lehrbuch der Differentialgleichungen, 2. Aufl., Vieweg, Braunschweig, 1912, pp. 105–128]. It is pointed out that $R=1$ leads to the ordinary WKB-approximation, that $R=t$ leads to the Airy-integral approximation [H. Jeffreys, Proc. London Math. Soc. (2) **23**, 428–436 (1924); R. E. Langer, Bull. Amer. Math. Soc. **40**, 545–582 (1934); Physical Rev. (2) **51**, 669–676 (1937)], which, by using the invariance of Schwarz' derivative with respect to linear transformations, may be improved to the Imai form [ibid. **74**, 113 (1948); **80**, 1112 (1950)], and that $R=\lambda-x^2$ would correspond to a case where the solution is essentially approximated by Hermite polynomials times an exponential function.

In order to improve the WKB-approximation, the same idea of comparing the various differential equations of the second order has now been independently developed by Miller and Good. They have particularly treated the potential-well and potential-barrier problems, where there are two turning points. The approximate solutions, expressed in Hermite polynomials times exponential functions, have the advantage that they are continuous across both turning points. The barrier-transmission problem is also treated uniformly for energies above and below the peak of the barrier. The approximate wave functions may be used for calculating matrix elements, and diagonal elements may sometimes be evaluated without numerical integration. A numerical example shows the great accuracy of the whole approach. *P. O. Löwdin* (Uppsala).

Lotkin, Mark. Some problems solvable on computing machines. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Memo. Rep. No. **693**, 15 pp. (1953).

The scope of present-day computational problems is sketched, and the qualities considered essential for a modern computing machine listed. Properties of the larger machines in use at the Ballistic Research Laboratories (ENIAC, EDVAC, ORDVAC, Bell Relay) are listed. The following types of problems considered at BRL are briefly discussed: solution of linear algebraic equations and matrix inversion; trajectory computations; calculation of transonic gas flow past a wedge; tabulation of cumulative binomial probabilities. *M. A. Hyman* (Washington, D. C.).

Goheen, Harry. A cryptographic machine. Proc. Iowa Acad. Sci. **60** (1953), 489–491 (1954).

Tomovich, Rajko. A versatile electronic function generator. J. Franklin Inst. **257**, 109–120 (1954).

Fucks, W. Über ein statistisches Gerät. Naturwissenschaften **41**, 57–58 (1954).

Bower, R. E., and Higman, J. H. The numerical solution of certain nonlinear supersonic flow problems on the IBM card programmed calculator. J. Aeronaut. Sci. **21**, 205–206 (1954).

Wotruba, Karel. Die Voraussetzungen der Lösung eines elektrischen Modells der elastischen Platte. Czechoslovak J. Phys. **2**, 56–63 (1953). (Russian. German summary)

***Belgrano, J. C., Lopez Nieto, A., y Urcelay, J. M. Tratado de nomografia.** [Treatise on nomography.] Instituto Tecnico de la Construccion y del Cemento, Madrid, 1953. xii+387 pp. 125 pesetas.

This is a comprehensive treatise on two types of nomogram: (i) charts consisting of families of curves in the plane read with and without a transparent sheet and (ii) alignment nomograms read with a straight edge. The theory and its application to a vast number of examples taken from engineering and other practice are discussed. The mathematical analysis is less nearly complete than in d'Ocagne's treatise [Traité de nomographie, 2ième éd., Gauthier-Villars, Paris, 1921], but results from the literature are frequently stated and explained in greater detail: an italicized theorem based principally on Gronwall's fundamental paper [J. Math. Pures Appl. (6) **8**, 59–102 (1912)] occupies nearly three pages. The means of recognizing whether empirical data can be represented by a nomogram and methods of constructing the nomogram when possible are discussed. The figures are plentiful but unsatisfactory because the paper is poor and the reduction of scale too great. The index lists only the examples under the names of eight fields in which they arise. A terse summary given in both English and French is a welcome feature. The book is a useful addition to the literature on the subject. It can be used as a text, but it is not run-of-the-mill and no exercises are included. *J. M. Thomas* (Durham, N. C.).

RELATIVITY

Finzi, Bruno. Su le equazioni di campo della teoria relativistica unitaria di Einstein. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 581–588 (1953).

In the most recent version of his unified theory, Einstein [The meaning of relativity, 4th ed., Princeton, 1953; these Rev. **14**, 805] imposes a priori the condition

(*) $$g^{\psi}{}_{\mu,\lambda}=0$$

upon the skew-symmetric part of the fundamental tensor-density, the remaining field-equations being obtained from a variational principle. This the author regards as unsatisfactory in a unified theory, inasmuch as the condition is a basic one that expresses the irrotational character of the electromagnetic tensor. Taking Einstein's own action-integral and applying a suitable variation, he obtains the Einstein field-equations. Another type of variation leads to field equations similar to those appearing in other unified theories. And finally, if the variation of g^{ψ} is taken to be the sum of a solenoidal variation and of one obtained from an infinitesimal variation of the space-time frame, then all the

field-equations of Einstein's theory are obtained, (*) included. *H. S. Ruse* (Leeds).

Udeschini, Paolo. Spostamento delle righe spettrali per effetto di un campo magnetico elementare nella nuova teoria relativistica unitaria di Einstein. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **16(85)**, 136–142 (1952).

The author considers the case of a weak field in Einstein's unified theory, weak in the sense that the asymmetric fundamental tensor is given by

$$g_{ik}=a_{ik}+b_{ik}+c_{ik},$$

where a_{ik} is galilean; b_{ik} is the (skew-symmetric) conjugate of the electromagnetic tensor, and is small of the first order compared with a_{ik}; and c_{ik}, which characterizes the gravitational field, is symmetric and small of the second order. [Cf. the author's paper in Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 390–394 (1951); these Rev. **13**, 169.] He shows how, in such a case, an electromagnetic field can affect the frequency of radiation through its gravitational action. Thus for a magnetic field of force due to a magnetic pole, the variation of frequency is proportional to the square of the intensity of the field.

H. S. Ruse (Leeds).

Bose, S. N. The affine connection in Einstein's new unitary field theory. Ann. of Math. (2) **59**, 171–176 (1954).

In Einstein's generalized relativity the equations

$$\frac{\partial g_{\mu\nu}}{\partial x^\sigma}=g_{\mu\lambda}\Gamma^\lambda_{\sigma\nu}+g_{\lambda\nu}\Gamma^\lambda_{\mu\sigma},$$

which relate the non-symmetric tensor $g_{\mu\nu}$ with the non-symmetric affine connection $\Gamma^\lambda_{\mu\nu}$, play a fundamental role. The author sketches a method for solving these equations for the Γ's as functions of the g's and their derivatives. The method involves using the eigenvalues and eigenvectors of the antisymmetric part of $g_{\mu\nu}$. *A. H. Taub* (Zurich).

Bose, S. N. Une théorie du champ unitaire avec $\Gamma_\mu\neq 0$. J. Phys. Radium (8) **14**, 641–644 (1953).

The author derives a new set of unified field equations from a variational principle involving a non-symmetric tensor $g_{\mu\nu}$ and a non-symmetric affine connection. The field equations satisfy Einstein's hermitian principle but do not satisfy the auxillary condition that the vector Γ_μ, the trace of the antisymmetric part of affine connection, vanishes. The author feels that the vector Γ_μ is related to the current vector of the electromagnetic field and also to the Lorentz ponderomotive force vector and that it should not vanish in a unified field theory. *A. H. Taub* (Zurich).

Bose, S. N. Certaines conséquences de l'existence du tenseur g dans le champ affine relativiste. J. Phys. Radium (8) **14**, 645–647 (1953).

The author determines the integrability condition of the equations

$$\frac{\partial g^{\mu\nu}}{\partial x^k}+g^{\mu\lambda}\Gamma^\nu_{k\lambda}+g^{\lambda\nu}\Gamma^\mu_{k\lambda}=0$$

for the functions $g^{\mu\nu}$. These equations occur in Einstein's generalized theory of relativity. *A. H. Taub.*

Callaway, Joseph. The equations of motion in Einstein's new unified field theory. Physical Rev. (2) **92**, 1567–1570 (1953).

It is shown that Einstein's generalized theory of relativity is such that singularities in the field do not obey the Lorentz equations of motion for a charged particle in an electromagnetic field. The method used is the quasi-static approximation method of Einstein, Infeld and Hoffmann [Ann. of Math. (2) **39**, 65–100 (1938)] applied to the field equations of the generalized theory of relativity. The calculation is carried out to the fourth order. *A. H. Taub.*

Scherrer, Willy. A propos des théories unitaires du champ. C. R. Acad. Sci. Paris **237**, 554–555 (1953).

Let $\psi_\alpha=\psi_{\alpha i}dx^i$ be a linear differential form, where $\psi_{\alpha i}$ are functions of the coordinates x^j $(i, j=1, 2, \cdots, n)$, transforming like a covariant vector in the index i. Any linear combination of such ψ_α is again an invariant linear differential form. Thus, in the set spanned by the ψ_α there is an interplay between coordinate and affine transformations analogous to that between coordinate and spinor transformations. The author raises the possibility of using the linear set of forms ψ_α, rather than riemannian space, as the underlying mathematical framework for a unitary field theory but makes little concrete progress in this direction.

A. J. Coleman (Toronto).

Schaffhauser-Graf, Edith. Versuch einer 4-dimensionalen einheitlichen Feldtheorie der Gravitation und des Elektromagnetismus. J. Rational Mech. Anal. **2**, 743–765 (1953).

Die Theorie basiert sich auf einer Finslerschen Geometrie, gegeben durch eine Funktion $L(x, u)$, homogen vom erster Ordnung in den Komponenten des Vektors u^i. Das kovariante Differential eines Vektors ξ^h ist

$$D\xi^h=d\xi^h+C^h_{ij}\xi^i du^j+\Gamma^h_{ij}\xi^i dx^j.$$

Die Cartanschen Bedingungen bestimmen die C^h_{ij} und Γ^h_{ij} eindeutig aus L. Der Fundamentaltensor ist $g_{ij}=\frac{1}{2}\partial_i'\partial_j'L^2$ $(\partial_i'=\partial/\partial u^i)$. Die Finsler-Geometrie hat drei Krümmungstensoren und den symmetrischen Tensor $A_{ijk}=LC_{ijk}$. Es wird nun vorausgesetzt, dass die Parallelverschiebung des Linienelementes integrabel ist. Das elektromagnetische Feld wird dargestellt durch $F_{ij}=\nabla_{[i}A_{j]}$ $(A_j=A_{j\alpha}g^{i\alpha})$. Die Maxwellschen Gleichungen sind dann automatisch erfüllt und auch ergibt sich der Erhaltungssatz der Ladung. Aus der Bianchischen Identität wird der Erhaltungssatz für Energie und Impuls der Materie erhalten. Die Felder sind vom Linienelement u^i, das die Geschwindigheit eines Elektrons entspricht, abhängig. Wenn das Feld F_{ij} verschwindet sind die Übertragungsparameter unabhängig von u^i und man erhält in diesem Falle die Einsteinschen Gravitationsgleichungen. *J. Haantjes* (Leiden).

Kustaanheimo, Paul. A note on the transformability of spherically symmetric metrics. Proc. Edinburgh Math. Soc. (2) **9**, 13–16 (1953).

It is well known that every spherically symmetric metric of relativity may be written in the form

$$ds^2=e^\nu dt^2-r^2(d\theta^2+\sin^2\theta\, d\varphi^2)-e^\lambda dr^2,$$

where ν, λ are functions of r and t. The author proves that this line element can be transformed into the isotopic form

$$ds^2=e^\nu dt^2-e^\mu(dr^2+r^2d\theta^2+r^2\sin^2\theta\, d\varphi^2),$$

where ν, μ are again functions of r and t. *M. Wyman.*

Bakamjian, B., and Thomas, L. H. Relativistic particle dynamics. II. Physical Rev. (2) 92, 1300–1310 (1953). [For part I see Thomas, Physical Rev. (2) 85, 868–872 (1952); these Rev. 13, 893.] A system of non-interacting particles is treated relativistically by performing a contact transformation to variables which are the relativistic generalizations of internal coordinates and momenta together with coordinates of the center of mass and the total momentum. Interaction may be introduced by replacing the mass by a function of the internal variables, and spin variables may be introduced as quasi-momenta. Hamiltonians derived by Darwin and by Breit for two particles are shown to be special cases of this formalism.

H. C. Corben (Pittsburgh, Pa.).

Hössjer, Gustav. On the postulates of electrodynamics. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 135–142 (1952).

The author first points out that the Maxwell equations of field-free space are equivalent to the constancy of two surface integrals in Minkowski space-time. He then discusses three topics: (1) an electron at rest, (2) a moving electron and (3) an electron in a small uniform electric field. The relation between the discussion of these three topics and the first observation is not clear. *A. H. Taub.*

Pratelli, Aldo M. Sopra i tensori spazio-temporali di Hertz e di Riesz per il campo elettromagnetico neutro. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 16(85), 449–464 (1952).

This paper is concerned with relationships between the various bivectors (6-vectors) of relativistic electromagnetism. In a flat space-time referred to a general coordinate-system, let $F_{\alpha\beta}$ be the bivector of electric force and magnetic induction, and $f_{\alpha\beta}$ be that of electric displacement and magnetic force. Let an asterisk denote the conjugate of a bivector (e.g., ${}^*f^{\alpha\beta}=\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}f_{\gamma\delta}$), and let a solidus denote covariant differentiation. Then the Maxwell-Lorentz equations in a region free from charge and current may be satisfied by putting

$$F_{\alpha\beta}=\Phi_{\beta/\alpha}-\Phi_{\alpha/\beta}, \quad {}^*f^{\alpha/\beta}=\psi^{\beta/\alpha}-\psi^{\alpha/\beta},$$

with $\Phi_{\alpha/}{}^{\alpha}=0$, $\psi^{\gamma}{}_{/\gamma}=0$, where Φ_α is the potential 4-vector and ψ^γ the 'antipotential' 4-vector. If $\Pi_{\alpha\beta}$ is the Hertzian bivector, then $\Pi_{\alpha\sigma/}{}^{\sigma}=\Phi_\alpha$, ${}^*\Pi^{\gamma\alpha}{}_{/\alpha}=-\psi^\gamma$ and $F_{\alpha\beta}+\Pi_{\alpha\beta/\sigma}{}^{\sigma}=f_{\alpha\beta}$. The bivector $r_{\alpha\beta}$ of M. Riesz [Acta Math. 81, 1–223 (1949); these Rev. 10, 713] satisfies

$$r_{\alpha\sigma/}{}^{\sigma}=\Phi_\alpha, \quad {}^*r^{\alpha\beta}{}_{/\beta}=0, \quad F_{\alpha\beta}=-r_{\alpha\beta/\sigma}{}^{\sigma}.$$

As $r_{\alpha\beta}$, $\Pi_{\alpha\beta}$ have the same divergence Φ_α, the tensor

$$s_{\alpha\beta}=r_{\alpha\beta}-\Pi_{\alpha\beta}$$

is solenoidal ($s_{\alpha\beta}{}^{\beta}=0$), while

$${}^*s^{\lambda\rho}{}_{/\rho}=\psi^\lambda, \quad f_{\alpha\beta}=-s_{\alpha\beta/\sigma}{}^{\sigma}.$$

If $M_{\alpha\beta}$ is the bivector of electric and magnetic polarization, so that, with suitable units, $M_{\alpha\beta}=F_{\alpha\beta}-f_{\alpha\beta}$, then $M_{\alpha\beta}=-\Pi_{\alpha\beta/\sigma}{}^{\sigma}$. Thus the tensors $\Pi_{\alpha\beta}$, $r_{\alpha\beta}$, $s_{\alpha\beta}$ are similarly related to $M_{\alpha\beta}$, $F_{\alpha\beta}$, $f_{\alpha\beta}$, respectively, the latter being minus the dalembertians of the former.

Various flux-theorems follow, the general nature of which may be sufficiently indicated by quoting just one of them, namely,

$$\oint_\sigma \Pi_{\alpha\beta}n^\beta d\sigma=\int_\tau \Pi_{\alpha\beta}{}^{/\beta}d\tau=\int_\tau \Phi_\alpha d\tau,$$

τ being a 4-dimensional region bounded by a hypersurface σ of unit normal vector n^α. *H. S. Ruse* (Leeds).

Takeno, Hyôitirô. A generalization of special Lorentz transformation in de Sitter space-time. Progress Theoret. Physics 10, 431–441 (1953).

The transformations of the full Lorentz group are generalized in de Sitter space-time from the stand-point of the group of motions. When the curvature of the space-time tends to zero, the generalized transformations tend to the well-known forms of the corresponding transformations of the Lorentz group. Some properties of the transformations, especially those of the generalized special Lorentz transformation, are studied. (Extract from the author's summary.) *A. G. Walker* (Liverpool).

Takeno, Hyôitirô. Theory of the spherically symmetric space-times. II. Group of motions. J. Sci. Hiroshima Univ. Ser. A. 16, 67–73 (1952).

Takeno, Hyôitirô. Theory of the spherically symmetric space-times. III. Class. J. Sci. Hiroshima Univ. Ser. A. 16, 291–298 (1952).

Takeno, Hyôitirô. Theory of the spherically symmetric space-times. IV. Conformal transformations. J. Sci. Hiroshima Univ. Ser. A. 16, 299–307 (1952).

Takeno, Hyôitirô. Theory of the spherically symmetric space-times. V. n dimensional spherically symmetric space-times. J. Sci. Hiroshima Univ. Ser. A. 16, 497–506 (1953).

Takeno, Hyôitirô. Theory of the spherically symmetric space-times. VI. Form-invariant tensors under group of motions and parallel tensors. J. Sci. Hiroshima Univ. Ser. A. 16, 507–523 (1953).

Takeno, Hyôitirô, and Ikeda, Mineo. Theory of the spherically symmetric space-times. VII. Space-times with corresponding geodesics. J. Sci. Hiroshima Univ. Ser. A. 17, 75–81 (1953).

Using the notation and abbreviations introduced in the first paper in this series [J. Math. Soc. Japan 3, 317–329 (1951); these Rev. 13, 985] the author continues his study of the general spherically symmetric space-time S_6, for which a coordinate system can be chosen so that the metric takes the form

$$ds^2=-A(r,t)dr^2-B(r,t)(d\theta^2+\sin^2\theta\,d\phi^2)+C(r,t)dt^2.$$

The present papers are mainly concerned with the classifications of S_6 according to various characteristic properties. In II, spaces are classified according to their groups of motions, of dimensionality 10, 7, 6, 4 or 3, and III is concerned with the "class" of S_6, i.e. the least r such that S_6 can be imbedded isometrically in $(4+r)$-dimensional flat space. It is already known that, for S_6, $r\leq 2$, and the present paper gives the conditions for $r=1$ and for $r=2$. In IV is examined the general conformal transformation which preserves spherical symmetry, and in V the results of the first four papers of the series are generalised to n-dimensional spherically symmetric space-time in which the metric is as above with $d\theta^2+\sin^2\theta\,d\phi^2$ replaced by the metric of a unit $(n-2)$-sphere.

Papers VI and VII return to 4-dimensional space-time. In VI are found those tensor fields which are invariant formally under the transformations of the groups discussed in II, and in particular the parallel tensor fields, i.e. those with vanishing covariant derivatives. In VII it is shown that geodesic-preserving correspondences may exist between certain spherically symmetric space-times.

A. G. Walker (Liverpool).

Takeno, Hyôitirô. Static spherically symmetric space-times in general relativity. Progress Theoret. Physics **10**, 509–517 (1953).

This paper continues the discussion of spherically symmetric space-times given in the papers reviewed above. The concept of 'staticness' is introduced as an intrinsic property of the space-time, and conditions are given in terms of invariants introduced in earlier papers. From the present point of view the de Sitter space-time is essentially static; other examples are considered in detail. *A. G. Walker.*

Tyabji, S. F. B. Equations of motion for continuous matter of special relativity in the Lagrange variables. Nature **172**, 1147–1148 (1953).

Dugas, René. Sur les pseudo-paradoxes de la relativité restreinte. C. R. Acad. Sci. Paris **238**, 49–50 (1954).

Sciama, D. W. On the origin of inertia. Monthly Not. Roy. Astr. Soc. **113**, 34–42 (1953).

The author describes a theory of gravitation which ascribes the inertia of a test particle to the relative motion of distant matter in the universe. The gravitational field is provisionally characterized by a four-vector potential which satisfies Maxwell's equations. A more complete and Lorentz-invariant theory is to follow in which the gravitational field will be described by a tensor potential. *A. E. Schild.*

***Synge, J. L. Geometrical mechanics and de Broglie waves.** Cambridge Monographs on Mechanics and Applied Mathematics. Cambridge, at the University Press, 1954. vi+160 pp. $4.75.

In optics there exist two mathematical theories, Hamilton's theory, starting from a variational principle, and Maxwell's theory. In this book the author develops a theory of relativistic geometrical mechanics by applying Hamilton's methods in the Minkowskian space-time. The theory is mathematically based upon a medium function $f(x^i, \alpha^i)$, homogeneous of the first degree in the components α^i of a timelike vector. Rays are curves in space-time satisfying the variational principle $\delta \int f ds = 0$ $(\alpha^i = dx^i/ds)$. A covariant vector p_i at a point determines a ray the direction α^i of which being given by $\partial f/\partial \alpha^i = \theta p_i$. The tangent spaces of a 3-space Σ_0 determine in this way a system of rays. If a constant value of the action $V = \int f ds$ is measured off along these rays a 3-space Σ is obtained. Σ represents the history of a 2-dimensional moving surface and is in fact a de Broglie wave. The group velocity of the waves is the ray velocity. The familiar formula for group velocity is obtained without any reference to wave length or frequency. In chapter III the author considers some medium functions which appear to be of physical interest. For a free particle: $f = mc(-\alpha_i \alpha_i)^{\frac{1}{2}}$. Possible focal mirrors are constructed. The function $f = mc(-\alpha_i \alpha_i)^{\frac{1}{2}} - iU\alpha_4 c^{-1}$ with a discontinuity in U furnishes the base for the study of refraction at a potential jump. For a particle in an electromagnetic field with 4-potential A_α the function $f = mc(-\alpha_\beta \alpha_\beta)^{\frac{1}{2}} - A_\beta \alpha_\beta$ is taken. In chapter IV the concept of phase is introduced by means of what is called a primitive quantization. It consists of assigning in a domain occupied by a system of de Broglie waves (determined by an initial wave Σ_0) a phase factor $\cos(V/h + \eta)$ where V is the characteristic function of the system and η is a constant. The distance between adjacent events of equal phase on a ray is called the absolute de Broglie wave-length. The relative wave length and frequency can be computed from this absolute wave length. A discussion of the quantization of a hydrogen atom leads to the usual fine-structure formula and to the Zeeman effect. The last chapter contains a generalization to spaces without metric. In this case the medium function defines a Finsler geometry. This theory is applied to problems involving two particles.

J. Haantjes (Leiden).

MECHANICS

***Artobolevskiĭ, I. I. Teoriya mehanizmov i mašin. [The theory of mechanisms and machines.]** 2d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 704 pp. 15.75 rubles.

This new edition of a 1945 textbook has been made necessary, according to the preface, by the progress in Soviet science, by new ideological directives of the party and the government, and by Stalin's "brilliant essay on 'Marxism and the questions of linguistics'." In the selection of the topics the book shows preferences similar to those encountered in other recent Russian books in the same field [Dobrovol'skiĭ, Theory of mechanisms, Gostehizdat, Moscow, 1951; these Rev. **13**, 995; Koževnikov, Theory of mechanisms and machines, Moscow, 1949; these Rev. **12**, 135]. Favored are structure and classification of mechanism, four-bar linkages, spur gears, and dynamics of machinery. Casual treatment is accorded to plane kinematics, linkage design (in spite of the recent and important Russian contributions), and spatial kinematics. In some cases the allotment of space to details has been influenced by the urge to present some recent Russian work (including that of the author).

The treatment of credits may be of some interest. Every effort has been made to connect old results, no matter how minor and how identified usually, with Russian names. The name index of 107 entries contains only eight Western names (Leonardo da Vinci, Bernoulli, Euler, Wittenbauer, etc., but not Reuleaux, Burmester, or Watt). Euler is invariably referred to as "a Russian academician". A component of mechanisms credited by Assur to Sylvester is credited to Assur. Moreover, these mentions of foreign scholars refer to historical surveys (of which there are three in the book). In the cases where the names are mentioned in connection with theorems (Euler-Savary, Roberts, Sylvester), the page is not shown in the name index (exception: Peaucellier, p. 53). There is no mention of Watt's mechanism, or of the Burmester design method [although it was fully presented by the author, Bloh, and Dobrovol'skiĭ in Design of mechanisms, Gostehizdat, Moscow-Leningrad, 1944; these Rev. **13**, 788]. The Russian credits are also erratic. Although the names of Levitskiĭ and Dimentberg are mentioned, this happens only enumeratively in the three surveys of Russian achievements. The important approximate design method of Levitskiĭ is given two pages, and the trail-blazing work of Dimentberg on spatial linkages is not presented at all.

Turning to technical matters, this review will concentrate on the section devoted to structure and classification of mechanisms, a problem of strange fascination to Russian writers. First of all, let us submit that the problem has no applied significance: if a mechanism has enough members to make its structure not obvious, its performance (i.e.,

motions or trajectories generated) can hardly be related to its structure in any useful manner because of the high order of the algebraic equations involved, and the large number of parameters appearing in their coefficients (it seems that the four-bar curves still offer more possibilities than we can utilize now). Let us then assume that the problem has purely cognitive importance, and that the main purpose of a classification is to define classes to whose elements the same formula for maximum (normal) mobility applies (singular conditions can always increase mobility).

Assur [Izv. Sankt-Peterburg. Politekh. Inst. **20**, 329–385 (1912); **21**, 581–635 (1913)] wrote 112 earnest pages on the subject, most reverently referred to in Soviet literature but little read or utilized. Assur recognized the dependence of his problem on the unsolved problem of the classification of topological graphs, and never claimed that his classification was exhaustive. The author, with very simple means, professes to solve the problem in all generality. Since his solution is very often mentioned in Russian writings, an account of it follows. There exists an "entirely general" mobility formula by Dobrovol'skiĭ: (1) $w=(6-m)n-\sum(k-m)p_k$, $k=5, \cdots, m+1$, where w is the degree of freedom of the mechanism with one bar fixed, $n+1$ the number of bars, p_k the number of pairs with relative mobility k ($k=5$ for a rotary or sliding pair), and m a magic number defining a "family" in the Artobolevskiĭ sense. It is easily seen that $m=3$ makes (1) valid for a plane mechanism. The reviewer has been unable to find a definition of m beyond the vague statement that "m is the number of constraints on the mechanism as a whole". Moreover, the definition of m cannot be gleaned from the examples given: the author simply announces the value of m for each case. Now, it is evident that (1) can be solved for m for any w, and that any constraints on the bars stem from the pairs (e.g., parallel hinge axes make the mechanism a plane one). Is it possible to determine m without studying the relations (possibly singular) between the hinges? This does not seem feasible to the reviewer, who finds the genealogy of Artobolevskiĭ's families very enigmatic. *A. W. Wundheiler.*

*Gres, W. H. **Die geometrischen Verhältnisse bei der Herstellung unregelmässiger Flächen. Geometrische Erzeugung und Nachformverfahren.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1953. vi+95 pp. DM 12.00.

The principal classes of techniques for producing complicated surfaces are the freehand methods, the pattern methods and the geometric generation methods. Only the last two are described in detail in this booklet. The pattern, which may be plane or solid, may be used to guide the cutting or shaping tool to give exact copies. Reversed or inverted shapes, or other affine transformations of the pattern, may be produced with the aid of simple linkage and pantograph mechanisms. The surfaces produced include cylindrical, conical and toroidal surfaces.

The geometric methods employ simple mechanisms to generate sinusoidal, cycloidal, trochoidal and involute cylinders and cones which find use as cams, spur gears and bevel gears. Helical and hyperboloidal surfaces are also used as gear shapes. Other complicated surfaces are obtained as envelopes of cutters and grinders of various shapes moved over a geometric path or guided by a pattern. The theoretical possibilities are limited by the practical and precision requirements that only a few elements be used in the geometric mechanism. *M. Goldberg.*

Tolle, O. Die Trägheitspolkurve der Schubstange eines zentrischen Schubkurbelgetriebes (Näherungslösung). Ing.-Arch. **21**, 365–367 (1953).

Federhofer's analytic expression for the locus of the centers of oscillation of the connecting-rod of a crank and crosshead mechanism [Österreich. Ing.-Arch. **5**, 240–245 (1951); these Rev. **13**, 173] is replaced by a graphical construction. The dimensions of the approximating ellipse are given. *M. Goldberg* (Washington, D. C.).

Kováč, Jozef. Contribution to the proof of Hartmann's theorem. Mat.-Fyz. Sborník Slovensk. Akad. Vied Umení **1**, 51–58 (1951). (Slovak. Russian summary)

In this paper the method of vector analysis is applied to prove Hartmann's theorem which is used for a simple construction of the center of curvature of the trajectory of a point of a rigid system in its motion in the plane.

Author's summary.

Dobrovol'skiĭ, V. V. Introduction to the dynamics of statically indeterminate mechanisms. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **8**, no. 30, 5–56 (1949). (Russian)

Geronimus, Ya. L. On the mass of a body reduced to the line of impact. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 631–633 (1953). (Russian)

The author proves first that the angular velocity imparted to a resting rigid body by a single-force blow along a line AB, and of a torque M_c about the mass center C, has the component M_c/Mp^2 normal to ABC (M=mass of body; p=radius of the central ellipsoid of gyration normal to ABC). [A complete determination of this velocity is given in Routh's Elementary rigid dynamics, 7th ed., Macmillan, London, 1905, Ex. 4, art. 310.] Since M, p are only two numbers, it is clear that the particle-velocity projections onto ABC can be duplicated if the body is reduced to two rigidly-connected particles. If one of them, P, is at B, ABC being a right angle, this reduction is clearly unique. The velocity acquired by P will be clearly the same as if it had been free. Its mass, by Joukowsky's definition, is the "body mass reduced to the line of blow". [There is considerable lost motion in this short paper.] *A. W. Wundheiler.*

Lovera, Piera. Sopra un problema dinamico studiato dal Volterra. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **86**, 100–106 (1952).

The author gives by means of vectorial methods simple proofs of some results of Volterra [same Atti **33**, 451–475 (1898)], who considered dynamical systems the velocities of which are linear homogeneous in a given set of parameters.

O. Bottema (Delft).

Schuh, Fred. Motion of an excentrically loaded sphere on a horizontal plane, in connection with the magic top "tippe top". Nederl. Akad. Wetensch. Proc. Ser. A. **56**=Indagationes Math. **15**, 423–432 (1953). (Dutch. French summary)

A rigid body, in the form of a sphere, has a distribution of mass with rotational symmetry about a diameter. The center of figure M and the center of mass Z are distinct. The author discusses the familiar problem of determining the motion of the body when the body is supported by a horizontal plane and the initial state of motion is that in which the body is rotating about the axis MZ, with Z directly below M. Certain conditions are found which are

necessary in order that the body may rotate into the position in which Z is directly above M. The stabilities of the motions in which Z is, respectively, directly below and directly above M are also discussed. *L. A. MacColl.*

***Gallissot, François. Sur une forme nouvelle des équations de la dynamique des systèmes matériels à un nombre fini de degrés de liberté.** Comptes Rendus du Congrès des Sociétés Savantes de Paris et des Départements tenu à Grenoble en 1952, Section des Sciences, pp. 21–33. Gauthier-Villars, Paris, 1952.

Les formes différentielles

$$f=\delta_{ij}(mdv^j-X^jdt)(dx^i-v^idt) \quad \text{(en algèbre ordinaire)}$$
$$\omega=\delta_{ij}(mdv^j-X^jdt)\wedge(dx^i-v^idt) \quad \text{(en algèbre extérieure)}$$

peuvent toutes les deux servir à engendre les équations du mouvement d'un point matériel. L'auteur préfère la forme ω. Si $d\omega=0$, ω est la dérivée extérieure de $\omega^*=mv_idx^i-Hdt$ engendrant l'invariant intégral de E. Cartan. Les équations du mouvement du point sont les équations associées à ω au sens de Cartan. Pour un système de coordonnées quelconques ρ^α ($\alpha=1,\cdots,6$) de position et de vitesse $\omega=k_{\alpha\beta}(d\rho^\alpha\wedge d\rho^\beta)-b_\alpha d\rho^\alpha\wedge dt$ et les équations du mouvement sont $\partial\omega/\partial(d\rho^\alpha)=0$. L'auteur montre qu'à tout système holonome à n degrés de liberté est associé une forme extérieure Ω de degré 2, dont les équations associées sont les équations du mouvement. Pour un système à n paramètres astreint à p liaisons compatibles on peut obtenir les équations du mouvement en adjoignant aux $2n-p$ équations associées à Ω p équations de Pfaff. *J. Haantjes.*

Dolmatov, K. I. Canonical equations of holonomic systems in redundant coordinates. Doklady Akad. Nauk UzSSR **1949**, no. 12, 3–7 (1949). (Russian)

This is, essentially, a special case of the matter in sec. 138 of Whittaker's "Analytical dynamics" [4th ed., Cambridge, 1937]: preservation of the Hamiltonian form under canonical transformations with simultaneous introduction of holonomic constraints. The author rederives this result crediting it to Suslov [Theoretical mechanics, OGIZ, Moscow-Leningrad, 1944].

In this paper, and in those of the two following reviews, a form of the equations of motion is used which Aržanyh announced under the name of the "new vortex principle of analytical dynamics" [Doklady Akad. Nauk SSSR (N.S.) **65**, 613–616 (1949); these Rev. **10**, 631] without actually stating it in the paper covered by the review just cited. If F is a function of t, q_i, $\dot q_i$, let $\partial_i'F$ denote its partial derivative with respect to q_i with t, $\dot q$ constant, and $\partial_t'F$ the derivative of F with respect to t with the q constant, but $\dot q$ regarded as functions of t only. Neither the notation, nor the definition, are those of Aržanyh but they result in validity of the "vortex equations",

$$\partial_t'p_i+\partial H/\partial q_i+(\partial_k'p_i-\partial_i'p_k)(\partial H/\partial p_k)=0.$$

The curl-like expressions are part of the Christoffel symbols, the other part being hidden in $\partial_t'p_i$. The merits of the addition of a new type of partial differentiation to the already existing confusion are not explained. *A. W. Wundheiler.*

Aržanyh, I. S. Characteristics of Suslov's method. Doklady Akad. Nauk UzSSR **1949**, no. 12, 8–11 (1949). (Russian. Uzbek summary)

The author is intrigued by the fact that relations of the type $F(t,q_i)=0$ [it should be $F(t,q_i)=$constant] may follow from Hamilton's equations. He verifies that the Hamilton-Jacobi theorem is still valid for such a singular system. *A. W. Wundheiler* (Chicago, Ill.).

Aržanyh, I. S. Generalization of the theorems of Jacobi and Liouville on integration of the canonical systems of Hamilton. Doklady Akad. Nauk UzSSR **1949**, no. 3, 3–5 (1949). (Russian. Uzbek summary)

Write F_i' for $\partial F/\partial q_i$, F_i'' for $\partial F/\partial p_i$, and F_t for $\partial F/\partial t$. The Hamilton-Jacobi theorem is generalized to allow incorporation of "quasi-holonomic" constraints of the form $dW_\sigma=0$. Let W, W_σ, λ_σ ($\sigma=1,\cdots,s<n$) be functions of t and q_i ($i=1,\cdots,n$) satisfying the partial differential equations (sum over repeated indices)

$$W_t'+\lambda_\sigma W'_{\sigma t}+H(t,q,p)=0, \quad p_i=W_i'+\lambda_\sigma W'_{\sigma i},$$
$$\lambda_{\sigma t}+\lambda'_{\sigma i}H_i''=0, \quad W_{\sigma t}+W'_{\sigma i}H_i''=0,$$

there being n constants of integration c_i. Then λ_σ, W_σ, $W_i^*=\partial W/\partial c_i+\lambda_\sigma\partial W_\sigma/\partial c_i$ will be constant during the motion.

A theorem of Liouville [Whittaker, Analytical dynamics, 4th ed., Cambridge, 1937, sec. 148] is generalized as follows, omitting the assumption that the known n integrals ϕ_i are in involution: If p_idq_i-Hdt takes the canonical form $dW+\lambda_\sigma dW_\sigma$, by virtue of the equations $d\phi_i=0$, then also λ_σ, W_σ, and W_i^* are constant. *A. W. Wundheiler.*

Aržanyh, I. S. A parametric form of the canonical transformations. Doklady Akad. Nauk UzSSR **1949**, no. 8, 8–11 (1949). (Russian. Uzbek summary)

The Pfaffian system (see the preceding review for notation)

$$dq_i=(H_{\mu i}''+F_{\mu\rho}H_{\rho i}')dt_\mu, \quad dp_i=-(H_{\mu i}'+F_{\mu\rho}H_{\rho i}')dt_\mu,$$
$$i=1,\cdots,n;\ \mu=1,\cdots,m;\ \rho=1,\cdots,\rho_m,$$

is invariant under the contact transformation

$$W_\sigma(t_\mu,q_i,p_i)=0, \quad p_i\delta q_i-\bar p_i\delta\bar q_i=\delta W(t_\mu,q_i,p_i)$$

if $\bar H_\mu=H_\mu+\partial W/\partial t_\mu+\lambda_\sigma\partial W_\sigma/\partial t_\mu$, $\bar H_{\mu\rho}=H_{\mu\rho}$, $\bar F_{\mu\rho}=F_{\mu\rho}$. There follows an application to a special case whose relevance is not discussed. *A. W. Wundheiler* (Chicago, Ill.).

Okabe, Jun-ichi. On the motion of a trolley-wire. Rep. Res. Inst. Appl. Mech. Kyushu Univ. **2**, 111–145 (1953).

This paper was inspired by the post-war electricity shortage in Japan. In the long wait for his turn on one of the too-infrequent streetcars, the author noted that the trolley wire began to vibrate against the dusky evening sky as the car approached, and contemplated the problem of the vibration of the trolley wire.

The transverse support wires are approximated by linearly damped springs periodically spaced along the main line. Between these support points the wire is assumed to satisfy a wave equation with a linear damping term added, and a non-homogeneous term is added which corresponds to the trolley load, assumed to be moving along the wire with constant velocity. The Fourier transform is used to solve the problem. By far the greater part of the author's labor lies in inverting the final transformed result to get the solution, and in making numerical calculations. The position of the support point as a function of time is graphed when the ratio of the street car velocity to the wave velocity in the wire is $\frac{1}{2}$, 1, and 2. *E. Pinney* (Berkeley, Calif.).

Hydrodynamics, Aerodynamics, Acoustics

Kil'čevskiĭ, N. A. On stress-, velocity- and density-functions in static and dynamic problems of the mechanics of continuous media. Doklady Akad. Nauk SSSR (N.S.) **92**, 895–898 (1953). (Russian)

If $T^{ij}=R^{ij}-\frac{1}{2}g^{ij}R^k{}_k$ ($i, j, k=1, \cdots, 4$), where R^{ij} is the contracted Riemann tensor based on the Riemannian metric tensor g_{ij}, then $T^{ij}{}_{,j}=0$. Let $g_{ij}=\delta_{ij}+\epsilon\Phi_{ij}$, where ϵ is a constant. Then $T^{ij}=\epsilon Q^{ij}_{(1)}+\epsilon^2 Q^{ij}_{(2)}+\cdots$, where the Q's are independent of ϵ. Then

$$T^{ij}{}_{,j}=0=\epsilon\partial Q^{ij}_{(1)}/\partial x^j+O(\epsilon^2) \quad \text{as } \epsilon\to 0,$$

so that $\partial Q^{ij}_{(1)}/\partial x^j=0$. Solve the equations

$$\|Q^{ij}_{(1)}\|=\begin{Vmatrix} \rho v_x^2-\sigma_x & \rho v_x v_y-\tau_{xy} & \rho v_x v_z-\tau_{xz} & \rho v_x \\ \rho v_x v_y-\tau_{yx} & \rho v_y^2-\sigma_y & \rho v_y v_z-\tau_{yz} & \rho v_y \\ \rho v_x v_z-\tau_{zx} & \rho v_y v_z-\tau_{zy} & \rho v_z^2-\sigma_z & \rho v_z \\ \rho v_x & \rho v_y & \rho v_z & \rho \end{Vmatrix}$$

for the quantities appearing on the right, and set $x^1=x$, $x^2=y$, $x^3=z$, $x^4=t$. These quantities satisfy the equations

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho v_x)+\frac{\partial}{\partial y}(\rho v_y)+\frac{\partial}{\partial z}(\rho v_z)=0,$$

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}=\rho\left[\frac{\partial v_x}{\partial t}+\frac{\partial v_x}{\partial x}v_x+\frac{\partial v_x}{\partial y}v_y+\frac{\partial v_x}{\partial z}v_z\right], \text{ etc.},$$

which are formally identical with the equation of continuity and dynamical equations of Newtonian continuum mechanics. *J. L. Ericksen* (Washington, D. C.).

Bjørgum, Oddvar. On Beltrami vector fields and flows ($\nabla\times\mathbf{v}=\Omega\mathbf{v}$). I. A comparative study of some basic types of vector fields. Univ. Bergen Årbok. Naturvit. Rekke **1951**, no. 1, 86 pp. (1952).

The author takes up the classical but little searched problem of determining, describing and characterizing "Beltrami fields," in which the curl is parallel to the field. As the author observes, some basic properties of these fields as well as their importance in classical hydrodynamics were first mentioned by Gromeka (1881) and Beltrami (1889) [Ist. Lombardo Sci. Lett. Rend. (2) **22**, 121–130], while their importance in gas dynamics, resting upon a theorem of the reviewer and R. Prim [U. S. Naval Ordnance Lab. Memo. **9416** (1947); these Rev. **11**, 623], was emphasized by Neményi and Prim [Proc. 7th Internat. Congress Appl. Mech., 1948, v. 2, pp. 300–314; these Rev. **11**, 473, and other papers, especially R. Prim, J. Rational Mech. Anal. **1**, 425–497 (1952); these Rev. **14**, 107]. The author's program is set forth in his introduction: ". . . a study of special vector fields, properly defined by differential equations and cultivated without any direct reference to applications, should prove as fruitful for problems expressed by vectors as has been the study of special functions for problems expressed by scalars, provided that no insurmountable mathematical difficulties are encountered at an early stage of the investigations suggested." This work is the first part of a comprehensive treatise on Beltrami fields, including all that is known of them and a number of results original with the author. This first part contains only general theorems.

Section 1 presents classical developments and some theorems of Neményi and Prim. In the case when the factor Ω in the defining relation $\nabla\times\mathbf{v}=\Omega\mathbf{v}$ is a constant rather than a general scalar, the author calls the field "Trkalian" [in the reviewer's opinion, "Gromeka field" would have been more just, since the 1881 treatise of Gromeka [Collected works, Izdat. Akad. Nauk SSSR, Moscow, 1952, pp. 76–148; these Rev. **15**, 89] deals intensively with the problem of determining such fields from appropriate boundary conditions].

Section 2 develops an intrinsic expression for the curl of a vector field:

$$\nabla\times\mathbf{v}=\mathbf{t}\Omega v+\mathbf{n}\frac{\delta v}{\delta b}+\mathbf{b}\left(-\frac{\delta v}{\delta n}+v\kappa\right),$$

where $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$ are unit vectors in the principal directions of the vector-line of $\mathbf{v}$, where δ denotes a directional derivative, and where κ is the curvature of the vector-line of $\mathbf{v}$. From this formula and from a generalization where $\mathbf{n}$ and $\mathbf{b}$ are replaced by any pair of unit vectors forming with $\mathbf{t}$ an orthogonal set follow in Section 3 characterizations of solenoidal, lamellar, complex-lamellar, and Beltrami fields in terms of their vector-lines and the variation of certain scalars along them. For example, necessary and sufficient conditions for a Beltrami field are

$$\frac{\delta\log v}{\delta n}=\kappa, \quad \frac{\delta v}{\delta b}=0.$$

In Section 4 the author in analogy to Kelvin's "complex-lamellar fields" defines more generally a "complex . . . vector field" as one which is everywhere proportional to a . . . vector field. Thus, for example, the "generalized Beltrami fields" which were introduced by Neményi and Prim are here called "complex Beltrami fields." Providing we allow the vector-magnitude to become infinite, the class of vector-lines of a . . . field is no less general than that for a complex . . . field. The author studies the possible types of vector-line pattern for the usual types of field. For a Beltrami field the result is rather complicated; as a special case it includes Neményi and Prim's theorem that the vector-lines of a Beltrami field are straight if and only if the magnitude of the field is uniform.

Section 5 concerns representations of Beltrami fields. For potentials of the type $\alpha\nabla\beta+\nabla\gamma$ or $\nabla\alpha\times\nabla\beta+\nabla\gamma$ the results are complicated. However, the author obtains the following interesting result: given a Beltrami field $\mathbf{v}$, we can choose co-ordinates q^1, q^2, q^3 such that the q^3-lines are the vector lines, $g_{13}=0$, $g_{23}=q'g_{33}$, $vg_{33}=1$; conversely, if a co-ordinate system is such that $g_{13}=0$, $g_{23}=q'g_{33}$, then the vector field tangent to the q^3-lines and of magnitude such that $vg_{33}=1$ is a Beltrami field.

Section 6 takes up boundary conditions. Most of the considerations here are incomplete because they assume that some or all of the scalar potentials α, β, γ will be single-valued, which cannot be expected in general. A particularly interesting result is that if γ in the representation $\alpha\nabla\beta+\nabla\gamma$ is single valued, then for a Beltrami field which does not cross a closed surface bounding a volume V we have

$$\int_V \Omega v^2 d\tau=0,$$

and hence Ω cannot be of one sign. Since as the author has shown earlier the value of Ω is a measure of the failure of the vector-lines to have a normal congruence, this result casts light on the vector-line pattern.

Section 7 concerns pseudoplane vector-fields, introduced by Euler and studied extensively by Berker [Sur quelques cas d'intégration des équations du mouvement d'un fluide visqueux incompressible, Taffin-Lefort, Paris-Lille, 1936]. These are of the type $\mathbf{v}\times\mathbf{k}=0$ ("first kind") or $\mathbf{v}=\mathbf{v}(x, y)$ ("second kind"), ordinary plane fields being characterized

by both these conditions together. Here the author collects known representations and studies the vector-lines, particularly in the case when they are straight.

This work deserves to be marked as the first to consider expressly an important new branch of analysis. While hydrodynamical interpretation motivates much of what is done, it is brought in only for examples of the results, which in all cases are purely vectorial. [See the following review.]

C. Truesdell (Bloomington, Ind.).

Bjørgum, Oddvar, and Godal, Thore. On Beltrami vector fields and flows ($\nabla\times\mathbf{v}=\Omega\mathbf{v}$). II. The case when Ω is constant in space. Univ. Bergen Årbok. Naturvit. Rekke 1952, no. 13, 64 pp. (1953).

This is a treatise on "Trkalian fields" [see the preceding review]. The first chapter concerns fluid flows whose velocity is Trkalian. Any such field is a solution of the hydrodynamical equations for incompressible fluids. For the inviscid case, the flow must be steady. For a viscous fluid of kinematic viscosity ν the flow must be of the form $\mathbf{v}=\exp(-\nu k^2 t)\mathbf{u}$, where $\mathbf{u}$ is a steady Trkalian field with $\Omega=k$ [Trkal, Časopis Pěst. Mat. Fys. 48, 302–311 (1919)]. The authors discuss superposability [cf. J. Strang, Communications Fac. Sci. Univ. Ankara 1, 1–32 (1948); these Rev. 11, 472] and the possibility of Trkalian oscillation and waves in heterogeneous inviscid fluids. While they mention Stekloff's method of obtaining Trkalian fields [Ann. Fac. Sci. Univ. Toulouse (2) 10, 271–334 (1908)], they prefer explicit series or potential representations. Their principal result (§§2.5–2.7) is that any Trkalian field has a representation

$$\mathbf{v}=k\nabla H\times\mathbf{e}+\mathbf{e}k^2H+\mathbf{e}\cdot\nabla\nabla H,$$

where $\mathbf{e}$ is a fixed unit vector and the "potential" H satisfies

$$\nabla^2H+k^2H=0,$$

a notable advance over Gromeka's equation $\nabla^2\mathbf{v}+k^2\mathbf{v}=0$. The various previously known Trkalian fields are easily derived.

From p. 30 onward the authors discuss special cases. There are many carefully drawn and detailed figures enabling the reader to visualize the nature of the flows which these fields can represent. Particularly interesting are some of the examples in cylindrical co-ordinates. In all cases illustrated boundary conditions appropriate to fluids are satisfied.

C. Truesdell (Bloomington, Ind.).

Popov, S. G. On the theory of vortical flows of an ideal incompressible fluid. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk 8, no. 3, 3–6 (1953). (Russian)

It is shown that a time-dependent plane flow with vorticity $-f_1(t)$ everywhere and body force $X=\frac{1}{2}yf_1'(t)$, $Y=-\frac{1}{2}xf_1'(t)$ has a stream function

$$\psi(x,y,t)=\tfrac{1}{4}y^2f_1(t)+\psi_1(x,y,t),$$

where ψ_1 is a harmonic function of x and y. Similarly, solutions of $\psi_{xx}+\psi_{yy}=\psi$ are stream functions of steady flows with $X=Y=0$ and vorticity $-\psi$. First integrals are found for both types of flows.

J. H. Giese.

Marini, Marino. Le azioni aerodinamiche su ali a freccia in moto oscillatorio. Aerotecnica 33, 275–287 (1953).

The author considers the forces on an oscillating arrowhead wing in incompressible flow. He extends the method of Weissinger [Math. Nachr. 2, 45–106 (1949); these Rev. 11, 64] and expresses the induced downwash due to his assumed vortex system as made up of the two-dimensional value plus an approximate correction term. The author treats in detail the singularities due to the "cranked" form of the lifting lines. No worked example is given, but at first sight the method looks promising, especially for pitching oscillations, when the calculations would be notably simpler than in the general case.

M. J. Lighthill.

Stepanov. G. Yu. Construction of double-rowed grids by the method of the hodograph of the velocity. Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 593–598 (1953). (Russian)

Suppose the z-plane contains two infinite grids of airfoils with the same complex period. Consider incompressible flow through the double grid with velocity $\bar{V}_1(\bar{V}_2)$ at infinity up-(down-)stream. The hodograph is infinitely many sheeted with logarithmic branch points at $\bar{V}_1$ and $\bar{V}_2$. A single period strip in the z-plane maps onto the interiors of the images L_1' and L_2' of the two airfoils, which lie on two sheets connected along a cut joining two branch points $\bar{V}_{p1}$ and $\bar{V}_{p2}$ of the first order. The transformation $\frac{1}{2}(\zeta+1/\zeta)=(2\bar{V}-\bar{V}_{p1}-\bar{V}_{p2})/(\bar{V}_{p1}-\bar{V}_{p2})$, combined with transformations of the type $\zeta_n=(\zeta-\zeta_0)^{1/k}$ if L_1' or L_2' has corners, maps this doubly connected region onto a topological annulus in the ζ-plane. This can finally be mapped between concentric circles in a v-plane by $\zeta=\zeta(v)$, and then by $u=\ln v$ onto a rectangle, a pair of opposite sides being maps of the grid airfoils, where the stream function is constant. The complex velocity potential function $W(u)$ is doubly periodic and must have logarithmic singularities at $u(\bar{V}_1)$ and $u(\bar{V}_2)$. $W(u)$ can be expressed in terms of logarithms of Weierstrass σ-functions. To construct explicit double grid flows the author suggests the inverse procedure of choosing $W(u)$ and $\zeta(v)$, and ultimately returning to the z-plane by $z=\int dW/\bar{V}$. Also considered is the case in which both grids extend to infinity downstream, in which a period strip maps onto a circle, and the analog of $W(u)$ merely involves logarithms.

J. H. Giese.

John, Fritz. Two-dimensional potential flows with a free boundary. Comm. Pure Appl. Math. 6, 497–503 (1953).

L'auteur décrit une méthode pour obtenir des solutions exactes des équations du mouvement à potentiel et avec ligne libre d'un liquide pesant dans un plan vertical. Le mouvement, qui peut ne pas être permanent, est donné sous forme paramétrique

$$z=f(w,t), \tag{1}$$

$$F=F(w,t), \tag{2}$$

w étant un paramètre complexe, z et F les arguments des plans du mouvement et du potentiel complexe. La ligne libre étant supposée correspondre à des valeurs réelles de w, l'auteur montre que, pour représenter un mouvement de l'espèce envisagée, $f(w,t)$ et $F(w,t)$ doivent satisfaire aux deux équations:

$$f_{tt}+ig=ir(w,t)f_w, \quad \text{pour } w \text{ réel}, \tag{3}$$

$$F_w=f_t(w,t)f_w(w,t), \tag{4}$$

où $r(w,t)$ est un coefficient qui est réel pour des arguments réels. Inversement, toute solution analytique de l'équation parabolique (3), où r est réel pour w réel, représente en variables de Lagrange, la ligne libre d'un mouvement du type considéré, pour lequel le potentiel complexe est donné par une quadrature au moyen de (4).

Lorsque le mouvement est permanent f et F peuvent s'exprimer à l'aide d'un seul paramètre complexe W, et les solutions sont alors définies par les deux équations différentielles:

$$f'(W)+ig=ir(W)\cdot f'(W), \tag{5}$$

$$F'(W)=\overline{f'(\overline{W})}\, f'(W). \tag{6}$$

Ceci est illustré par l'exemple suivant, obtenu en prenant $r(W)=c$. On a alors

$$z=\frac{g}{c}W+Ae^{icW},\quad F=\left(\frac{g^2}{c^2}+A^2c^2\right)W+\frac{2gA}{c}\cos cW$$

où c et A sont deux constantes réelles et positives. La ligne libre est une cycloïde ($A<g/c^2$) d'amplitude A et de longueur d'onde $L=2\pi g/c^2$. Les lignes de courant ont ces mêmes éléments de symétrie et la même périodicité géométrique que la ligne libre et elles varient dans le même sens que celle-ci. Lorsqu'elles sont suffisamment voisines de la surface, elles n'ont pas de recoupement et elles peuvent alors être matérialisées par une paroi sur laquelle s'effectue l'écoulement. Pour A/L petit la profondeur est grande devant A et L, et la paroi est sensiblement rectiligne, ses oscillations étant petites par rapport à celles de la surface. On a là un exemple explicite des écoulements à périodicité géométrique dont l'existence a été établie dans le cas général par R. Gerber [C. R. Acad. Sci. Paris **233**, 1261–1263 (1951); ces Rev. **13**, 594]. *R. Gerber* (Toulon).

Zarantonello, Eduardo H. A constructive theory for the equations of flows with free boundaries. Collectanea Math. **5**, 175–225 (1952).

Birkhoff, Young and Zarantonello have given an effective numerical method for the computation of plane flows with free boundaries, in particular those in which a cavity is formed by a free jet or infinite stream impinging on a curved obstacle [Proc. Symposia Appl. Math., 1951, v. 4, McGraw-Hill, New York, 1953, pp. 117–140; these Rev. **15**, 258]. In their work the flow problem is reduced to an equivalent functional equation [derived first by Levi-Cività, Rend. Circ. Mat. Palermo **23**, 1–37 (1907)] for which approximate solutions are obtained by a process of discretization and subsequent iteration. The present paper provides theoretical justification for this method. It is shown that, if the obstacle is convex to the flow and has not too large angular extent, then (1) the iteration is convergent, and (2) the solutions of the discrete equations tend to a solution of the given equation. For obstacles concave to the flow similar, but less general, results are given. For application of the results to the theory of conformal mapping one may refer to the paper cited above.

J. B. Serrin (Cambridge, Mass.).

Plesset, M. S. On the stability of fluid flows with spherical symmetry. J. Appl. Phys. **25**, 96–98 (1954).

The author points out that a recent analysis by Binnie [Proc. Cambridge Philos. Soc. **49**, 151–155 (1953); these Rev. **14**, 594] of the stability of a spherical interface between two fluids of different densities is in error. The problem is reconsidered by the author. A linearized analysis of a perturbation of the spherical surface leads to a differential equation for the time factor in the perturbation. Stability of solutions of this equation is briefly discussed, particularly in two special cases. Contrary to Binnie's conclusions, there are significant differences between the plane case considered by Taylor [paper cited in review of Binnie] and the spherical case. *J. V. Wehausen* (Providence, R. I.).

Poincelot, P. Application de la notion de vitesse de groupe à la propagation de la houle en eau profonde. Ann. Géophysique **9**, 158–160 (1953).

Un train d'ondes sinusoïdales commençant à l'origine des temps est représenté par une intégrale et son évolution est étudiée en fonction du parcours. Diverses transformations mettent en évidence une quantité h, qui caractérise l'écart entre le régime qui se propage effectivement et le régime permanent qui se propage avec la vitesse de groupe. Dans le cas de la houle en eau profonde cette quantité h diminue rapidement avec le parcours pour s'annuler à l'infini, ce qui montre que la notion de vitesse de groupe est dans ce cas rigoureuse à l'infini, et d'une bonne approximation dans la réalité. *R. Gerber* (Toulon).

Ursell, F. Short surface waves due to an oscillating immersed body. Proc. Roy. Soc. London. Ser. A. **220**, 90–103 (1953).

In an earlier paper [Quart. J. Mech. Appl. Math. **2**, 218–231 (1949); these Rev. **11**, 65] the author treated the problem of small vertical oscillations of a circular horizontal cylinder (radius $=a$), half submerged in an ideal fluid. Although the method of that paper produced a solution valid (the proof is completed in this paper) for all values of the parameter $N=\sigma^2a/g$ where σ is the frequency, it was suitable for computing the vertical force and wave amplitude at infinity only for small values of N. In the present paper the author uses a different method which provides a useful solution for large values of N. The method consists essentially in altering the integral equation for the velocity potential ϕ (which one obtains from the use of Green's theorem with ϕ and the Green function) in such a way as to obtain a new integral equation with a kernel of norm less than one. An iterative method may then be applied to obtain the solution as a series. The first two terms of the series are found explicitly. *J. V. Wehausen* (Providence, R. I.).

Keller, Joseph B., and Weitz, Mortimer. Reflection and transmission coefficients for waves entering or leaving an icefield. Comm. Pure Appl. Math. **6**, 415–417 (1953).

En utilisant la transformation de Green l'auteur calcule les coefficients de réflexion et de transmission pour une houle en présence d'un champ glaciaire superficiel, dans les deux cas d'une longueur d'onde grande ou petite par rapport à la profondeur. Les résultats de l'auteur coïncident avec ceux déjà connus (Weitz et Keller [mêmes Comm. **3**, 305–318 (1950); ces Rev. **12**, 762], Keller et Goldstein [Trans. Amer. Geophys. Union **34**, 43–48 (1953); ces Rev. **14**, 810], et Shapiro et Simpson [ibid. **34**, 36–42 (1953)]).

R. Gerber (Toulon).

*Gerber, Robert. **Sur les écoulements plans à potentiel des liquides pesants.** Comptes Rendus du Congrès des Sociétés Savantes de Paris et des Départements tenu à Grenoble en 1952, Section des Sciences, pp. 13–18. Gauthier-Villars, Paris, 1952.

This is essentially another brief statement of results announced in C. R. Acad. Sci. Paris **233**, 1261–1263 (1951); **235**, 693–694, 1601–1602 (1952); these Rev. **13**, 594; **14**, 508.

J. V. Wehausen (Providence, R. I.).

Hunt, J. N. A note on gravity waves of finite amplitude. Quart. J. Mech. Appl. Math. **6**, 336–343 (1953).

The author corrects a number of errors in Struik's paper [Math. Ann. **95**, 595–634 (1926)] demonstrating the existence of periodic waves in water of finite depth. The

errors seem chiefly to stem from one incorrect sign and do not vitiate the proof but do make some of the approximate formulas incorrect, e.g., that for the wave velocity.

J. V. Wehausen (Providence, R. I.).

Tanaka, Kiyoshi. On the sea-waves. Tech. Rep. Osaka Univ. **3**, nos. 55–73, 113–123 (1953).

In the first part of this paper the author derives from formal manipulation of a power series for the velocity potential an expression for the wave velocity which retains terms through the third order in the ratio wave-height/wavelength. He remarks that his result disagrees with that of Struik [Math. Ann. **95**, 595–634 (1926)]. However, his result also disagrees with that in the paper reviewed above because at one step he disregards some third-order terms.

In the second part the author treats diffraction of waves passing through a gap in an infinite breakwater and around a straight barrier. An approximate answer is found by combining known exact results for a semi-infinite breakwater. A separate treatment is given for the case when the wave length is large compared with the gap. [See Penney and Price, Philos. Trans. Roy. Soc. London. Ser. A. **244**, 236–253 (1952); these Rev. **13**, 790.] *J. V. Wehausen.*

Kawaguti, Mitutosi. Numerical solution of the Navier-Stokes equations for the flow around a circular cylinder at Reynolds number 40. J. Phys. Soc. Japan **8**, 747–757 (1953).

A viscous flow around a circular cylinder at Reynolds number 40 is calculated numerically by relaxation methods. According to the size of the meshes, the flow domain is subdivided into three sub-domains: (i) ordinary points, (ii) surface points and (iii) infinite points. The following results are typical. (1) The calculated flow pattern is remarkably close to that observed by experiment at Reynolds number 33.5, which has two symmetrically placed elongated vortices in the closed wake. This seems to show that a steady flow is theoretically still possible at Reynolds number 40. (2) The constant-vorticity and constant-velocity lines are drawn; this gives a good physical picture of the flow. The lines of constant unit velocity and of maximum vorticity and the boundary of the wake are of special interest. The author conjectures that all these lines at high Reynolds number will coalesce to a line corresponding to the separated free streamline in the boundary-layer theory. (3) Pressure distribution over the axis of symmetry and over the cylinder is calculated. The surface pressure compares very well with that measured by experiments at Reynolds numbers 36 and 45. The fact that the surface pressure in this range of Reynolds numbers depends only slightly on Reynolds number is surprising. (4) Good agreement with experiments is also accomplished in the calculation of drag.

Y. H. Kuo (Ithaca, N. Y.).

Dean, W. R. Note on the motion of viscous liquid past a parabolic cylinder. Proc. Cambridge Philos. Soc. **50**, 125–130 (1954).

The author finds an approximate solution for the steady two-dimensional motion of viscous incompressible liquid past a fixed parabolic cylinder. The parabolic coordinate system is used. In the limiting case of the flat plate, the expression for the stream function agrees with that found earlier by Carrier and Lin [Quart. Appl. Math. **6**, 63–68 (1948); these Rev. **9**, 477]. *C. C. Lin.*

Trilling, L. The boundary layer on a quarter infinite flat plate. Quart. Appl. Math. **12**, 80–83 (1954).

The plate is arranged so that the stream direction is at 45° to the edges. As the author suggests, other angles of approach can be obtained by stretching one of the coordiates. The fluid is incompressible. Using polar coordinates with origin at the leading corner, it is shown that the solution found by the reviewer (and others) for a yawed semi-infinite plate holds everywhere. This solution states that the flow is everywhere in the stream direction and along each streamwise line is given by the Blasius solution for plane flow. The resulting violation of the assumptions of the boundary layer equation along the streamwise axis through the leading corner is pointed out.

A more general investigation was reported by F. K. Moore [NACA Tech. Note no. **2279** (1951); these Rev. **12**, 872], who showed that the present result applies to flat plates of arbitrary leading-edge shape, in compressible or incompressible flow, and also noticed the violation of the assumptions along the streamwise lines through every leading-edge corner. *W. R. Sears* (Ithaca, N. Y.).

Holtsmark, J., Johnsen, I., Sikkeland, T., and Skavlem, S. Boundary layer flow near a cylindrical obstacle in an oscillating, incompressible fluid. J. Acoust. Soc. Amer. **26**, 26–39 (1954).

A viscous oscillating flow about a fixed circular cylinder is considered both theoretically and experimentally. By taking a simple unsteady solution of the linearized Navier-Stokes equations, satisfying the required conditions at infinity as the basic approximation, a first approximation to the exact solution is calculated. The solution shows that in the neighborhood of the cylinder there is a pair of vortices in each quadrant, rotating in opposite directions. The cores of the outer vortex systems will be at finite distance if the flow field is bounded; and will be at infinity if the flow field is unbounded. The theoretical predictions were verified very well by experiments conducted in a Kundt tube. By comparison of the potential velocity with the measured values, a boundary layer is defined, whose thickness decreases as the Reynolds number, based on the frequency of the oscillations and the radius of the cylinder. A curious phenomenon is observed where the actual velocity exceeds the corresponding potential value in a range of angles greater than 15° from the stagnation point. *Y. H. Kuo.*

Witting, Hermann. Über zwei Differenzenverfahren der Grenzschichttheorie. Arch. Math. **4**, 247–256 (1953).

The problem is to solve Prandtl's differential equations for steady plane laminar flow in the boundary-layer of an incompressible fluid. Schröder [Zentrale für wissenschaftliches Berichtswesen der Luftfahrtforschung des Generalluftzeugmeisters (ZWB), Forschungsbericht Nr. 1741 (1943) =NACA Tech. Memo. no. **1317** (1952); Math. Nachr. **4**, 439–467 (1951); these Rev. **13**, 879] and Görtler [Ing.-Arch. **16**, 173–187 (1948); these Rev. **10**, 336] have replaced differential equations by difference equations in somewhat different ways. The author asserts that both methods give answers of the same exactness, but that Schröder's method involves more effort and gives oscillating results which must be "smoothed". Both methods give poor results near the boundary, especially near the separation-point; the author proposes a method to overcome this difficulty in the paper reviewed below. *M. A. Hyman* (Washington, D. C.).

Witting, Hermann. Verbesserung des Differenzenverfahrens von H. Görtler zur Berechnung laminarer Grenzschichten. Z. Angew. Math. Physik **4**, 376–397 (1953).

The method of differences developed by H. Görtler [Ing.-Arch. **16**, 173–187 (1948); these Rev. **10**, 336] for the calculation of laminar boundary layers of plane, steady and incompressible flow functions in the neighbourhood of the point of separation holds only with a reduced degree of exactness. The same holds for the method of K. Schröder [Math. Nachr. **4**, 439–467 (1951); these Rev. **13**, 879]. This defect is overcome by means of a polynomial approximation of the velocity distribution in the neighbourhood of the wall. In addition, the sensitivity of the method with respect to propagation of errors is generally reduced. The relatively slight increase in labour is more than offset by the increased degree of exactness. The splitting phenomena inherent in the above-mentioned methods may be avoided by means of a suitable smoothing process. The method has been tried with success on several examples. (From the author's summary.) *C. C. Lin* (Cambridge, Mass.).

Berman, Abraham S. Laminar flow in channels with porous walls. J. Appl. Phys. **24**, 1232–1235 (1953).

The author finds a type of exact solution of the Navier-Stokes equation appropriate to the two-dimensional flow through a channel with two equally porous walls. An ordinary differential equation has to be integrated. This is carried out by the perturbation method. *C. C. Lin.*

Wasow, Wolfgang. On small disturbances of plane Couette flow. J. Research Nat. Bur. Standards **51**, 195–202 (1953).

The Orr-Sommerfeld equation for the stability of parallel flows is analyzed in detail for the case of plane Couette flow. A fundamental system of solution is found whose asymptotic behavior as $\alpha R \to \infty$ can be calculated for the whole complex y-plane (α is the wave number of the disturbance, R is the Reynolds number of the basic flow, and y is the independent variable of the differential equation). A disturbance with given α is proved stable, for any α, if αR is sufficiently large. In addition, an investigation is made of the asymptotic properties of an allied differential equation for which there are two regular inviscid solutions and yet an inner friction layer does exist, in contradistinction to the Couette case. *C. C. Lin* (Cambridge, Mass.).

Lock, R. C. Hydrodynamic stability of the flow in the laminar boundary layer between parallel streams. Proc. Cambridge Philos. Soc. **50**, 105–124 (1954).

The theory of small wavy oscillations in parallel flows is developed for the steady laminar flow of a horizontal wind over the surface of the liquid at rest, taking into account viscosity, gravity, and surface tension. In the gas phase, the method of analysis is that often used for other stability problems. In the liquid phase, further simplification can be obtained due to the fact that the fluid is nearly at rest. It is shown that there are two fundamental types of oscillation of the system, which may be called water waves and air waves. Curves showing the conditions for neutral stability of these two types of waves are given for a range of wind speeds from 100 to 300 cm./sec. Certain features shown by the results obtained are in general agreement with observation. The limitations of the theory are pointed out, especially the requirement that the air flow must be laminar. *C. C. Lin* (Cambridge, Mass.).

Haque, S. M. A. On the stability of the motion of a viscous liquid flowing between two parallel plates. Pakistan J. Sci. Res. **4**, 17–19 (1952).

The author claims to have shown that the plane Poiseuille motion is stable with respect to two-dimensional infinitesimal disturbances, but there seems to be a mistake in the basic equation used. Many misprints make a thorough check difficult. *C. C. Lin* (Cambridge, Mass.).

Bloom, Martin. On the calculation of laminar boundary-layer stability. J. Aeronaut. Sci. **21**, 207–210 (1954).

Lin, C. C. On a perturbation theory based on the method of characteristics. J. Aeronaut. Sci. **21**, 202–203 (1954).

v. Krzywoblocki, M. Z. The "independent scalars" in homogeneous turbulence in compressible media. J. Phys. Soc. Japan **8**, 745–747 (1953).

L'auteur étend aux fluides compressibles un ensemble de résultats de N. Coburn [Amer. J. Math. **74**, 296–306 (1952); ces Rev. **14**, 217] relatifs aux tenseurs de corrélation dans un fluide incompressible. N. Coburn considère les "tenseurs de corrélation" d'ordre $1, 2, 3, \cdots$ du champ de vitesses aléatoire, en turbulence homogène, en deux points séparés par le vectuer ξ^β. Il suppose que ces tenseurs dépendent soit seulement du vecteur ξ^β (isotropie), soit du vecteur ξ^β et d'un autre vecteur j^β (axisymétrie), soit du vecteur ξ^β et de deux autres vecteurs j^β, i^β (cas général). Il donne l'expression des tenseurs de corrélation à l'aide du nombre minimum de fonctions scalaires, compte tenu de la condition d'incompressibilité. L'auteur reprend la question lorsque les divergences de ces tenseurs, au lieu d'être nulles, sont des quantités tensorielles données, dépendant de fonctions scalaires données. Par exemple, pour un tenseur du second ordre $R^{\lambda\mu}$, on a $R^{\lambda\mu}{}_{,\lambda} = F_2(r,t)\xi^\mu$, $F_2 \neq 0$, et aux expressions données par Coburn s'ajoute le terme $\int r F_2 dr$. *J. Bass.*

Burgers, J. M., and Mitchner, M. On homogeneous non-isotropic turbulence connected with a mean motion having a constant velocity gradient. I, II. Nederl. Akad. Wetensch. Proc. Ser. B. **56**, 228–235, 343–354 (1953).

The statistical theory of turbulence in the absence of a mean flow is generalized to include a constant velocity gradient. In the first part of the paper, the basic equations are derived both in the correlation form and in the spectral form. It is possible that in an unlimited field with a constant velocity gradient the length scale of the turbulence would become infinite, so that the spectrum might be divergent at the lower end. No clear indication of such behavior was obtained from the equations. Part II of this paper deals with steady shear flow. By introducing Heisenberg's formula for the transfer function, the spectrum at large wave numbers is found to be of the form

$$E(k) = Ck^{-5/3}[1 + (k/k_s)^4]^{-4/3},$$

where k is the wave number and C and k_s are constants of integration, which are determined in terms of certain physical quantities. Finally, a brief discussion is given for the case of non-homogeneous turbulence. [Reviewer's remark: Similar work has been done by F. B. Reis in his Ph.D. thesis at Mass. Inst. Tech., 1952 (unpublished). Reis also demonstrated the permanence of the large eddies in the presence of a velocity gradient, assuming that the spectrum does not become divergent at the lower end. He also obtained the above formula for the spectrum by similar

methods and gave similar discussions of the physical interpretation of the constants.] *C. C. Lin.*

Lorenz, Edward N. The interaction between a mean flow and random disturbances. Tellus **5**, 238–250 (1953).

Statistical hydrodynamics is applied to the problem of the interaction between a mean flow and a superposed disturbance, in a two-dimensional homogeneous incompressible nonviscous fluid. The ensemble of all disturbances which may individually be superposed upon a given mean flow is assumed to be random, in the sense that it is unaltered if each disturbance is subjected to a change of sign, a translation in space, or a rotation. It is found that, ensemble-average-wise, kinetic energy is transferred from the disturbances to the mean flow if the mean flow is of small variance and coarse detail and the disturbances are on the average of large amplitude and fine detail, while kinetic energy is transferred in the opposite direction if the opposite situation exists.

This result is applied to the problem of the maintenance of kinetic energy in the earth's atmosphere against the dissipative effect of friction. There is some evidence that both the total kinetic energy and the kinetic energy of the mean flow can be maintained through the addition of new disturbances which form random ensembles, but that they can be maintained more efficiently, and probably are maintained, by the addition of new disturbances with a systematic lack of randomness. (From the author's abstract.)

C. C. Lin (Cambridge, Mass.).

Bjørgum, Oddvar. On energy transformations and the effect of fluctuating pressure gradients in turbulent shear flow. Univ. Bergen Årbok. Naturvit. Rekke **1951**, no. 3, 18 pp. (1952).

Bjørgum, Oddvar. On the steady turbulent flow along an infinitely long smooth and plane wall. Univ. Bergen Årbok. Naturvit. Rekke **1951**, no. 7, 14 pp. (1952).

Bjørgum, Oddvar. On the analogy between turbulent transfer of heat and momentum. Univ. Bergen Årbok. Naturvit. Rekke **1951**, no. 10, 8 pp. (1952).

In these three papers, the author reviews and comments on some of the familiar concepts in the theory of turbulence. Some modifications are suggested and compared with experiments. The first paper deals with energy transfer in turbulent shear flow through the work done by the pressure fluctuation and convection. The author claims to have shown that the former is zero. The second paper deals with the motion near a wall by means of various possible assumptions. The third paper emphasizes the effect of local pressure gradients on momentum transfer, which was used a long time ago by Taylor as a possible reason for preferring the vorticity transfer theory to the momentum transfer theory.

C. C. Lin (Cambridge, Mass.).

Kraichnan, Robert H. The scattering of sound in a turbulent medium. J. Acoust. Soc. Amer. **25**, 1096–1104 (1953).

The author investigates the scattering of a sound wave by a turbulent flow of low root-mean-square Mach number. The directional distribution of the scattered sound is found in a complicated form for a general turbulent field, and a conveniently simple form for isotropic turbulence. The results agree with those of the reviewer [Proc. Cambridge Philos. Soc. **49**, 531–551 (1953); these Rev. **15**, 365].

M. J. Lighthill (Manchester).

***Modern developments in fluid dynamics. High speed flow.** Edited by L. Howarth. 2 vols. Oxford, at the Clarendon Press, 1953. Vol. I: xvi+pp. 1–475 (9 plates); vol. II: viii+pp. 477–875 (17 plates). 84 shillings; $17.00.

Tous ceux qui connaissent l'importance des deux premiers volumes "Modern developments in fluid dynamics" [Oxford, 1938] par le professeur S. Goldstein avec la collaboration de nombreux savants Britanniques accueilleront avec joie les deux présents volumes consacrés aux fluides compressibles. Le professeur L. Howarth assume cette fois la difficile tâche d'éditeur en chef et présente deux volumes denses susceptibles de rendre les mêmes services pour les développements modernes consacrés aux grandes vitesses et aux effects de la compressibilité. Ces livres deviendront vite indispensables à tous ceux qui dans les années à venir s'orienteront vers l'étude des fluides, et l'on peut prédire sans risque que l'on s'y référera pendant longtemps. Avant de commencer cette analyse il faut dire tout d'abord que ces deux volumes constituent une preuve éclatante de l'éminente place tenue par les savants Britanniques dans les études modernes consacrées aux fluides; la plupart des auteurs qui ont écrit les chapitres composant ces ouvrages sont de notoriété internationale et font partie indiscutablement des meilleurs spécialistes mondiaux des sujets traités. Notons encore quelques qualités générales fort appréciables pour une telle "somme": la présentation quasi-parfaite des volumes, l'homogénéité des définitions, symboles et notations, l'équilibre des divers sujets qui donne déjà par un simple coup d'oeil sur la table des matières une idée de la place de chaque chapitre dans l'ensemble de l'édifice. De ceci il faut louer le professeur Howarth et les membres du "Fluid Motion sub-Committee of the Aeronautical Research Council" qui ont assuré la supervision de l'oeuvre entreprise.

L'ouvrage commence par une introduction d'une trentaine de pages d'un caractère élémentaire; le but poursuivi semble être de dégager une vue intuitive et simple des principaux thèmes développés dans la suite—écoulements subsoniques et supersoniques, ondes de Mach et ondes de choc, couche limite et transfert de chaleur—et de les présenter dans une perspective synthétique. L'étude proprement dite commence donc avec le chapitre II, consacré aux équations générales des écoulements dans les gaz, qui comme le précédent est rédigé par L. Howarth. Comme dans la suite du livre, de fréquentes références aux livres de Goldstein permettent de soulager l'exposé et de souligner les effets propres de la compressibilité. La délicate question de l'application des résultats de la thermodynamique aux écoulements des fluides est abordée avec beaucoup de soin et de précision; la difficulté essentielle qui consiste à appliquer des résultats valables pour des systèmes en équilibre à des phénomènes qui ne le sont pas est parfaitement dégagée. L'auteur montre en particulier comment on peut surmonter au moins partiellement cette difficulté en faisant appel aux notions de la théorie cinétique des gaz.

Le chapitre III (R. E. Meyer) est consacré à l'exposé de la méthode des caractéristiques. La théorie commence d'emblée avec le cas où l'entropie est variable d'une trajectoire à l'autre; l'exposé est fait pour un système de coordonnées curvilignes orthogonales arbitraire. Le présent développement reprend essentiellement les mémoires de l'auteur [Quart. J. Mech. Appl. Math. **1**, 196–219, 451–469 (1948); ces Rev. **10**, 338, 492]. A notre gré, l'importance des coordonnées caractéristiques, si magistralement mise en évidence par Meyer lui même [Philos. Trans. Roy. Soc.

London. Ser. A. 242, 153–171 (1949), p. 169; ces Rev. 11, 473], n'est pas assez soulignée. On trouve dans ce chapitre des applications et des exemples à propos desquels sont exposées les méthodes numériques et graphiques utilisées par les ingénieurs pour la résolution des problèmes pratiques. Le quatrième chapitre (C. R. Illingworth et G. J. Kynch) traite des ondes de choc. Après une introduction consacrée au concept d'onde, les équations des chocs sont établies dans le cas où aucun changement chimique n'est présent. Suivent quelques développements consacrés à la structure de l'onde de choc, aux apparitions et aux interactions élémentaires des chocs dans les écoulements stationnaires à deux dimensions, puis dans les écoulements non stationnaires dépendant d'une variable spatiale, spécialement dans le cas des ondes sphériques. Le chapitre V (W. G. Bickley) est consacré aux solutions exactes des écoulements stationnaires bidimensionnels: tourbillon, source, écoulement spiral, détente de Prandtl-Meyer, onde simple, écoulement supersonique au voisinage d'un profil polygonal, écoulement à symétrie axiale autour d'un cone de révolution. Les divers aspects de la théorie des écoulements à une dimension font l'objet du chapitre suivant (O. A. Saunders): influence de la variation de l'aire des sections du tube, du frottement, des échanges de chaleur. Les résultats théoriques sont comparés à ceux de l'expérience.

Le chapitre VII (M. J. Lighthill) est consacré à l'étude des équations des écoulements stationnaires à deux dimensions dans le plane de l'hodographe. Dans cette méthode, l'équation fondamentale est une équation linéaire dont les solutions à variables séparées s'expriment avec des fonctions hypergéomètriques (fonctions de Chaplygin), mais la formation de solutions présentant les singularités voulues et le passage au plan physique sont des problèmes difficiles. Le présent chapitre repose essentiellement sur les travaux de l'auteur [Proc. Roy. Soc. London. Ser. A. 191, 323–341, 341–351, 352–369 (1947); ces Rev. 9, 350, 391; 11, 870] concernant l'étude des profils et des tuyères. On y trouve également l'étude des lignes singulières de la transformation de l'hodographe (lignes limites et lignes de branchement). Le chapitre suivant (G. N. Ward) traite des méthodes approchées pour l'étude des écoulements stationnaires. Une soixantaine de pages pour un si vaste sujet comprennant les relations de similitude du type Prandtl-Glauert, les méthodes de Janzen et Rayleigh, un court paragraphe sur l'analogie électrique et toute l'aérodynamique supersonique linéarisée (ailes, fuselages, corps cylindriques) traitée d'une part par la méthode des sources en faisant appel à la théorie de l'intégrale prise en partie finie et d'autre part par la méthode des écoulements coniques. Le chapitre se termine par quelques pages sur les relations de similitude de von Kármán pour les écoulements transsoniques. L'étude beaucoup plus complexe de l'aérodynamique non stationnaire est abordée au chapitre suivant (G. Temple); écoulement supersonique et subsonique, écoulement autour d'une aile d'envergure infinie et d'une aile d'envergure finie, écoulements autour des fuselages. On trouvera également quelques comparaisons avec l'expérience dans le cas des mouvements à deux dimensions.

Le chapitre X (A. D. Young) est consacré à la théorie de la couche limite; une centaine de pages sont consacrées au sujet ce qui permet un exposé assez détaillé, tenant compte des différentes méthodes d'approche qui ont été proposées dans la littérature spécialement pour la couche limite laminaire. Quelques pages rendent compte des études faites sur l'interaction des ondes de choc avec la couche limite. Les chapitres suivants (XI, XII, XIII) sont consacrés aux méthodes expérimentales et à la discussion des résultats d'expérience (plus de 250 pages). Un dernier chapitre (H. B. Squire) assez développé traite des théories modernes relatives aux phénomènes de transfert de chaleur en écoulement laminaire et turbulent.

Nous avons assez dit tout le bien que nous pensons de cet ouvrage unique en son genre, pour qu'on nous permette certaines remarques critiques destinées à guider l'éventuel lecteur; qu'il soit entendu que ces remarques ne changent rien à l'excellente opinion d'ensemble que nous avons formulée. Le principal reproche est le suivant: la plupart des articles ont été écrits entre 1947 et 1949 alors que la date d'édition est 1953. Ce fait explique le paradoxe que nous avons déjà noté: dans de nombreux cas les auteurs n'ont pas pu tenir compte des résultats de leurs propres travaux parus après 1950, travaux qui à l'heure actuelle apparaissent parfois beaucoup plus importants que ceux dont on nous rend compte dans ces ouvrages. Ce retard est d'autant plus regrettable qu'en 1948 de nombreux chapitres de la dynamique des gaz aux grandes vitesses étaient en plein développement alors que les années 52–53 marquent une époque de plus grande stabilité. Le second reproche que nous ferons concerne la bibliographie et les références d'ordre historique; à notre avis, ce livre s'inspire trop exclusivement des seuls travaux Britanniques. La littérature soviètique (celle que l'on trouve dans les bibliothèques universitaires) est pratiquement complètement ignorée; et pourtant soit pour les méthodes de l'hodographe et les écoulements transsoniques (F. Frankl), soit pour l'approximation linéaire en supersonique stationnaire ou non stationnaire (Krassiltchikova), soit même pour la couche limite, les savants de ce pays ont produit d'importants travaux, devançant même parfois leurs collègues occidentaux. Mais même la littérature Américaine mériterait parfois un peu plus de considération. Un seul exemple: le livre de Courant et Friedrichs [Supersonic flow and shock waves, Interscience, New York, 1948; ces Rev. 10, 637] dont existaient des copies dactylographiées en Angleterre depuis 1945 est cité très peu de fois. Pourtant dans de nombreux cas les solutions proposées dans ce livre sont souvent plus intéressantes et les idées mises en relief ont parfois une résonance plus moderne. La place nous manque pour donner quelques exemples d'inexactitudes d'ordre historique, qui sont d'autant plus regrettables qu'elles seront authentiquées par des ouvrages par ailleurs excellents et propagées dans l'enseignement pendant de nombreuses années.

S'il fallait risquer un jugement d'ensemble, ce n'est ni l'éditeur en chef, ni les auteurs que nous mettrions en cause, mais la formule même d'un tel ouvrage. Ce n'est pas un cours accessible aux étudiants; et d'autre part chaque auteur manque de la place nécessaire pour donner un développement harmonieux avec les commentaires éclairants qui feraient pénétrer le lecteur dans leurs pensées et leurs vues; cette exiguïté de la place réservée à chacun explique aussi sans doute une rédaction parfois un peu rapide et les quelques petites négligences que nous avons relevées. Une chose nous parait certaine: si grande soit la valeur d'un tel ouvrage, elle reste nettement inférieure à la somme des réputations de chacun des auteurs. En refermant ces deux livres, on se prend à rêver: une monographie de 200 pages sur la méthode de l'hodographe écrite par Lighthill, une autre sur l'approximation linéaire écrite par Ward, une autre par Young sur la couche limite, etc., le résultat serait certainement merveilleux.

P. Germain.

McVittie, G. C. Spherically symmetric solutions of the equations of gas dynamics. Proc. Roy. Soc. London. Ser. A. **220**, 339–355 (1953).

The author applies the general integrals he has previously obtained [Quart. Appl. Math. **11**, 327–336 (1953); these Rev. **15**, 175] of the equations of conservation of mass and momentum for a perfect fluid to the case of spherically symmetric gas motions. It is shown that in this case the pressure, density and velocity of the gas involve one arbitrary function. The cases where this function is of the form $\varphi = f(t)w(rt^{-n})$ and the velocity is proportional to the radial distance are discussed in detail and various restrictions are imposed successively. Thus adiabatic motions are first discussed, then these are restricted further to satisfy the boundary conditions holding either at a contact surface or at a shock front. *A. H. Taub* (Zurich).

Kaplan, Carl. Incompressible flow past a sinusoidal wall of finite amplitude. NACA Tech. Note no. **3069**, 26 pp. (1954).

The author continues his earlier work [NACA Tech. Note no. **2383** (1951); **2748** (1952); these Rev. **12**, 875; **14**, 425] on hydrodynamical flow past a wavy wall. In this case the flow is incompressible. An inverse approach proves most convenient; the physical plane is expanded as a series in terms of the complex potential. The coefficients in the expansion must be determined implicitly from expansions which arise from the boundary conditions. The latter are solved heuristically by power series developments in two different parameters. The labor involved increases rapidly with the degree of the expansion. The author concludes that an integral equation procedure is better and converts his original expansion formally into an integral equation. He does not emphasize the fact that the equation is singular—this may prove a trap for the unwary. Others using the same method may prefer to short-cut the original expansion, obtaining the integral equation directly. The author takes advantage of the Bessel solution to the Kepler eccentric anomaly problem to obtain a better-than-first approximation to the solution of the integral equation. He states that it is not feasible to go farther in a completely analytical manner, and suggests a method for proceeding numerically by a successive approximation procedure. The convergence of this procedure is not established. *E. Pinney.*

Gotusso, Guido. Correnti a velocità sonica. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **16(85)**, 116–122 (1952).

The author discusses inconclusively whether in a plane flow of compressible fluid the fluid speed might everywhere equal that of sound. *M. J. Lighthill* (Manchester).

Giese, J. H., and Cohn, H. Canonical equations for nonlinearized steady irrotational conical flow. Ballistic Research Laboratories, Aberdeen Proving Ground, Md. Memo. Rep. No. **692**, 19 pp. (1953).

Les écoulements coniques sont caractérisés par le fait que le vecteur vitesse en un point M de l'espace ne dépend que de deux paramètres; ces deux paramètres définissent la direction de la demi-droite OM, O étant le sommet de l'écoulement conique. Dans l'espace de l'hodographe, l'écoulement a donc pour image une surface et la détermination d'une telle surface dépend de la résolution d'une équation aux dérivées partielles à deux variables. Les auteurs de ce rapport étudient le cas des écoulements irrotationnels; les équations dont dépendent ces écoulements sont données sous une forme très générale. Dans le cas où le problème est hyperbolique, il est indiqué de prendre des variables caractéristiques; dans le cas elliptique un système de variables isothermes. Le cas classique des écoulements de révolution supersoniques appartient à ce dernier cas. Une représentation conforme permet de définir la solution dans une couronne circulaire, l'une des cercles représentant l'obstacle conique, l'autre l'onde de choc, mais les conditions aux limites qu'il faut satisfaire rendent très difficile la solution du problème. Des méthodes numériques sont suggérées pour l'étude du problème ainsi posé. *P. Germain.*

Imai, Isao. Extension of von Kármán's transonic similarity rule. J. Phys. Soc. Japan **9**, 103–108 (1954).

Le présent article est le développement d'une communication présentée au Congrès International de Mécanique d'Istambul en 1952. Le but que se propose l'auteur est de fournir des règles de similitude pour des écoulements autour de profil minces, qui soient valides aussi bien dans le cas subsonique que dans le cas supersonique. Pour les écoulements transsoniques, ces règles doivent redonner les résultats de von Kármán. La bibliographie de cet article étant un peu sommaire, peut-être ne sera-t-il pas inutile de comparer à quelques uns des nombreux travaux consacrés à cette question. J. R. Spreiter [NACA Tech. Note no. **2273** (1951); ces Rev. **12**, 766] a donné des formules tenant compte des termes du premier ordre au voisinage de l'obstacle; les règles de D. C. Pack et S. I. Pai [Quart. Appl. Math. **11**, 377–384 (1954); ces Rev. **15**, 264] tiennent compte du terme non linéaire du second ordre qui donne des effets du premier ordre loin de l'obstacle; le présent article semble inclure les termes du second ordre au voisinage de l'obstacle. Notons enfin que l'auteur ne nous offre pas seulement une méthode pour aboutir aux règles de similitude, mais une méthode de calcul qui semble intéressante; des applications en seront publiées prochainement. *P. Germain.*

Gilbarg, D., and Paolucci, D. The structure of shock waves in the continuum theory of fluids. J. Rational Mech. Anal. **2**, 617–642 (1953).

Les auteurs essaient de justifier l'application des méthodes du continu à l'étude des ondes de chocs. En reprenant les solutions des équations de Navier-Stokes pour les gaz monoatomiques, les auteurs montrent que les valeurs prévues par la théorie sont en une assez bonne concordance avec les valeurs empiriques pour l'Helium et l'Argon. Il devait en être de même pour les gaz polyatomiques.

On peut encore améliorer les résultats en introduisant une loi de viscosité non linéaire. Les auteurs obtiennent, par exemple, pour une loi quadratique une très bonne concordance dans le cas d'Helium. *M. Kiveliovitch.*

Ting, Lu. Diffraction and reflection of weak shocks by structures. J. Math. Physics **32**, 102–116 (1953).

The pressure field due to the reflection and diffraction of a weak plane shock striking at normal incidence a rigid two-dimensional rectangular structure resting on a plane surface is found to the first approximation in shock strength, i.e., the shock is treated as a sound pulse. The problem is treated by the method of successive substitutions in that separate problems are considered for suitably chosen time intervals and the disturbance pressure for the nth time interval is written in the form $P = \sum_{i=1}^{n} S_i$. The time intervals are chosen so that each interval starts at the beginning

of a new diffraction regime and then the problem for each interval can be treated by techniques such as those developed by Evvard, Ward, and Keller and Blank.

For the case of an infinitely thin barrier the time intervals are of equal duration, h/C_0, where h is the height of the barrier and C_0 the speed of sound in the undisturbed region. The problem for each time interval can be classed as one of three different types and explicit solutions are given for the first three time intervals. For these intervals the total force on the thin barrier as a function of time is obtained and it is interesting to note that by the end of the third time interval this force is only 15% of the original force on the structure. Since for greater values of time the total force is less than this figure and approaches zero, this indicates that for simple structures a reasonable estimate of significant pressure and force distributions can be obtained by solving the problems for a relatively few time intervals.

Similar considerations are made for a rectangular barrier of length L and height h. The basic time intervals are derived from values of time $t_{mn}=(mL+nh)/C_0$, where m and n are positive integers. In this case there are also three typical problems, though it should be noted that this is strictly true only if L and h are incommensurate. It should also be noted that the method developed in this paper is actually applicable to any convex polygonal-shaped two-dimensional body, but other than the cases studied by the author, and a triangular-shaped body, the computational labor would undoubtedly be prohibitive. *P. Chiarulli.*

Miles, John W. Virtual momentum and slender body theory. Quart. J. Mech. Appl. Math. **6**, 286–289 (1953).

In an earlier paper [J. Aeronaut. Sci. **19**, 280–281 (1952)] the author extended Ward's linearized theory of supersonic flow past slender pointed bodies to include unsteady flow. The dipole field at infinity obtained from this solution is compared with that obtained by means of the virtual momentum approach. In this way it is shown that the transverse force on any local cross-section as given by the former method is in fact equal to the rate of change of the virtual momentum associated with the passage of a cylinder of this cross-section with the prescribed velocity of the body through a fluid at rest. *D. C. Pack* (Glasgow).

Morris, Deane N., and Smith, John W. The compressible laminar boundary layer with arbitrary pressure and surface temperature gradients. J. Aeronaut. Sci. **20**, 805–818 (1953).

In this paper, the authors proposed to solve the compressible laminar-boundary-layer problem by the Kármán-Pohlhausen method with great generality, such as arbitrary pressure and surface temperature gradients and arbitrary Prandtl number. By representing the velocity and temperature profiles by fourth and sixth degree polynomials respectively, integrations have been carried out for the following cases: (1) retarded and accelerated main stream flows of incompressible fluids with constant wall temperature; (2) flat plate with parabolic surface temperature at Mach numbers 0.5 and 3 and Prandtl number 0.72. As a practical application, the case of a biconvex circular-arc airfoil is considered for Mach number 2 and Prandtl number 0.72. [The reviewer must admit that he cannot fully follow the author's formulation of the initial conditions, in particular, the equation (28).] *Y. H. Kuo* (Ithaca, N. Y.).

Sorokin, V. S. A variational method in the theory of convection. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 39–48 (1953). (Russian)

L'auteur étudie un fluide dont le réchauffement n'est pas uniforme. En général le fluide ne peut pas être en équilibre étant donné que la différence de la densité des particules qui ne sont pas également chauffées provoque une convection. L'équilibre peut être atteint seulement exceptionnellement. Dans ce cas on observe que la convection apparaît brusquement, d'une façon discontinue, lorsque le gradient de la température atteint une certaine valeur critique. L'auteur étudie en détail ce cas intéressant en utilisant le calcul des variations. *M. Kiveliovitch* (Paris).

Yaglom, A. M. The dynamics of large-scale processes in a barotropic atmosphere. Izvestiya Akad. Nauk SSSR. Ser. Geofiz. **1953**, 346–369 (1953). (Russian)

L'auteur étudie le système linéarisé des équations d'un fluide barotrope en remplaçant dans ces équations les composantes de la vitesse et de la pression par leurs moyennes suivant l'altitude. Comme première approximation, l'auteur étudie en détail le cas d'une "terre plate". On obtient deux types de solutions dont le premier contient les solutions qui correspondent à un mouvement ondulatoire avec un champ stationnaire incompressible, et le second représente les perturbations qui se propagent avec une vitesse du son. Ce sont surtout les premières solutions qui présentent un grand intérêt en météorologie. Il s'agit donc d'éliminer les solutions du second type. L'auteur expose à cet effet deux méthodes: l'une qui consiste à remplacer l'équation de continuité par l'équation d'incompressibilité, et la seconde, la méthode de Kibel, qui remplace toutes les fonctions par des séries de fonctions développées suivant les puissances entières croissantes d'un très petit paramètre. En introduisant la sphéricité de la terre, la première méthode fournit les solutions obtenues par Blinova, et la seconde méthode permet d'obtenir les corrections lorsque l'on rejette la condition d'incompressibilité. *M. Kiveliovitch* (Paris).

Kitkin, P. A. Action of the wind on a body of water in a shallow closed basin. Doklady Akad. Nauk SSSR (N.S.) **91**, 1325–1328 (1953). (Russian)

L'auteur utilise les équations linéarisées de l'hydrodynamique avec deux coefficients différents pour l'échange turbulent horizontal et vertical. Pour résoudre le système l'auteur décompose les fonctions inconnues suivant les sinus et cosinus des arguments $\alpha_n x+\beta_n y$, x et y étant les coordonnées du plan. Les coefficients étant des fonctions de l'altitude et du temps. En substituant les séries ainsi formées dans les équations des mouvements et en égalant les coefficients ayant le même argument on obtient un système d'équations aux dérivées partielles de second ordre, du type parabolique. Pour résoudre ces équations l'auteur utilise la transformation de Laplace et obtient de cette façon une solution formelle du problème. *M. Kiveliovitch* (Paris).

Kitkin, P. A. On the dynamics of sea and ocean flows. Doklady Akad. Nauk SSSR (N.S.) **92**, 293–296 (1953). (Russian)

L'auteur forme les 3 équations du mouvement en partant des équations d'Euler. Les équations obtenues diffèrent des équations Navier-Stokes pour un fluide visqueux par l'asymétrie du tenseur des tensions. Afin de former un système complet d'équations, l'auteur ajoute aux trois équations du mouvement l'équation de continuité et d'in-

compressibilité. Malheureusement la solution de ce système complet présente des difficultés presque insurmontables.

M. Kiveliovitch (Paris).

Frankl', F. I. On the theory of motion of suspended sediment. Doklady Akad. Nauk SSSR (N.S.) **92**, 247–250 (1953). (Russian)

Cette question, assez difficile, n'a pas encore été étudiée suffisamment. Dans la présente note l'auteur se propose en partant des équations du mouvement des milieux continus de déduire un système d'équations de base. L'auteur arrive à former un système non complet, composé de deux équations de continuité et de six équations du mouvement.

M. Kiveliovitch (Paris).

Vâlcovici, Victor. Sur une théorie des alluvions. Acad. Repub. Pop. Române. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **4**, 653–660 (1952). (Romanian. Russian and French summaries)

Afin d'expliquer la suspension des particules solides dans un courant fluide laminaire, l'auteur détermine les forces hydrodynamiques qui agissent sur la surface d'un cylindre immobile situé dans un courant de vitesse horizontale linéairement croissante avec la hauteur. De cette manière, il trouve que le fluide exerce une force d'ascension Y dont l'expression

$$Y=2a\rho\left(a\int_C y^2dx-\int_C yqdx\right)$$

est mise sous la forme d'une formule Blasius-Tchaplyguine.

From the author's summary.

Elasticity, Plasticity

Brdička, Miroslav. The equations of compatibility and stress functions in tensor form. Czechoslovak J. Phys. **3**, 36–52 (1953). (Russian. English summary)

The author derives the Beltrami-Michell equations and some of the better known general solutions of the equations of linear elasticity, then expresses them in tensorially invariant form. *J. L. Ericksen* (Washington, D. C.).

Angelitch, T. P. Eine Bemerkung zu den Gleichungen von Beltrami-Michell. Acad. Serbe Sci. Publ. Inst. Math. **5**, 1–4 (1953).

The author attempts to translate the usual derivation of the Beltrami-Michell equations into tensor notation in general coordinates. The attempt is a failure. On the last line of p. 2 he adds together two zero tensors of different variance; as might be expected, this soon leads to errors: the quantity he singles out in line five of p. 3 is not a tensor except in rectangular Cartesian systems, nor is his final result (13) a tensor equation. *C. Truesdell.*

Bordoni, Piero Giorgio. Sopra le trasformazioni termoelastiche finite di certi solidi omogenei ed isotropi. Univ. Roma. Ist. Alta. Mat. Rend. Mat. e Appl. (5) **12** (1953), 237–266 (1954).

This paper is devoted to the study of a special strain energy. The author gives certain heuristic reasons for wishing stress-strain relations of the type

$$t^i_j=[(p-ne^k_k)\delta^i_j-2me^i_j]f,$$

where p, n, and m are functions of the actual temperature and the temperature in the state of ease and f is a function of the principal invariants of the spatial strain tensor e^i_j. The author determines f so that Finger's conditions for existence of a strain energy shall be satisfied [cf. §41 of the reviewer's paper in J. Rational Mech. Anal. **1**, 125–171, 173–300 (1952); **2**, 593–616 (1953); these Rev. **13**, 794; **15**, 178]. The remainder of the paper concerns the properties of this special elastic body: relation of m and n to the isothermal Lamé constants, inequalities for these latter, determination of the corresponding thermodynamic potential, sign of the second temperature derivative of various quantities, hydrostatic stress, simple extension. The author concludes that the behavior of this body is in good accord with experiments on metals. [For an account of the other special theories of elasticity, see Chapter IV C of the reviewer's paper cited above.] *C. Truesdell* (Bloomington, Ind.).

Bordoni, Piero Giorgio. Deduzione di un'equazione di stato dei solidi dalla teoria delle trasformazioni termoelastiche finite. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 784–790 (1953).

In the theory defined by his special strain energy in the paper reviewed above, the author obtains the equation of state for finite spherical extension. He discusses various physical coefficients which follow from this theory. The compressibility comes out to be the same as in the infinitesimal theory. For Grüneisen's constant the author obtains $\frac{1}{2}+\lambda/\mu$, a value of the right order for some metals but not for others. *C. Truesdell* (Bloomington, Ind.).

Ledinegg, E., und Urban, P. Zur Bestimmung der elastischen Konstanten isotroper Festkörper mittels Ultraschall. Acta Physica Austriaca **8**, 16–27 (1953).

The authors determine the equations for the proper frequencies for dilatational waves in two problems concerning a concentric cylindrical cavity in a fixed cylindrical rod of elastic material, supposed in a state of radial and longitudinal extension only. In both these the cavity is filled by another body of different elastic constants. In the former, that body is another cylinder; in the latter, a cylindrical plate. The authors compare their results in limiting cases with those they had obtained earlier by perturbation methods [same Acta **7**, 420–435 (1953); these Rev. **15**, 181]. *C. Truesdell* (Bloomington, Ind.).

Pailloux, Henri. Nouvelles applications du calcul fonctionnel à la mécanique. Ann. Sci. Ecole Norm. Sup. (3) **70**, 1–49 (1953).

The author applies the method described in a previous paper [same Ann. (3) **69**, 213–257 (1952); these Rev. **14**, 816] to the statics and dynamics of plates and shells and also to problems of a more technical engineering nature involving systems of beams, such systems being replaced by continua. There is also an application to hydrodynamics. The object is to obtain differential equations rather than to solve them, and the method (which is essentially an energy method) may be described by considering the equilibrium of a shell, omitting body forces and surface forces (they are included in the paper, which also deals with dynamics). The strain energy is an integral involving the first derivatives of the displacement (u, v, w), and the condition that this integral shall be stationary gives, of course, the usual equations of elastic equilibrium. But instead of using a general displacement the author takes (*) $u=u_1+\zeta u_2$, $v=v_1+\zeta v_2$, $w=w_1+\zeta w_2$, where $(u_1, \cdots, w_2)$ are functions of (x, y) and ζ is the distance of the point in question from

the mean surface of the shell, measured parallel to the z-axis. This restriction on the displacement is justified by the thinness of the shell. The strain energy for (*) is an integral of the form $\int\int\cdots dxdy$, since ζ disappears on integration across the thickness of the shell, and when the stationary condition is applied to this integral, we get partial differential equations for $(u_1, \cdots, w_2)$ with (x, y) as independent variables. Thus the method is one in which approximate equations are obtained from an exact variational principle by restricting the functions involved to a certain type.

J. L. Synge (Dublin).

Hwang, Keh-Chih. General elastic theory of thin plates and shells with small deflections. Acad. Sinica Science Record 5, 87–124 (1952). (Chinese summary)

Led on by what he considers errors in the work of E. Reissner [Amer. J. Math. 63, 177–184 (1941); these Rev. 2, 272] and the reviewer [Trans. Amer. Math. Soc. 58, 96–166 (1945); these Rev. 7, 231], the author considers the small deformation of thin shells. By a somewhat more lengthy analysis the author derives anew the equations of R. Byrne [Univ. California Publ. Math. (N.S.) 2, 103–152 (1944); these Rev. 5, 250], of whose work he is apparently unaware. The "error" in Reissner's work is a difference of opinion regarding the appropriateness of various possible expansions; that in the reviewer's work seems to be a misunderstanding arising from the reviewer's having used conventional notations in an unconventional sense. The author is not aware of the prior work of Novozhilov [C. R. (Doklady) Acad. Sci. URSS (N.S.) 38, 160–164 (1943); see also "Foundations of the nonlinear theory of elasticity," Gostehizdat, Moscow-Leningrad, 1948, chap. VI; these Rev. 5, 139; 12, 651] nor of the critical (but incomplete) review by Hildebrand, Reissner, and Thomas [NACA Tech. Note no. 1833 (1949); these Rev. 11, 69]. Mention should also be made of Reissner's latest equations [J. Math. Physics 31, 109–119 (1952); these Rev. 13, 1006]; these include those of Byrne and the author as a special case. In the reviewer's opinion, all work on this subject (including his own) is purely formal, and the various results obtained by different perturbation processes cannot be shown to be right or wrong by the a priori arguments always employed. What is lacking is a mathematical theorem making precise the status of solutions of any given set of proposed equations with respect to corresponding solutions of the three-dimensional theory. *C. Truesdell* (Bloomington, Ind.).

Zerna, W. Berechnung von Translationsschalen. Österreich. Ing.-Arch. 7, 181–187 (1953).

A surface of translation is swept out when a plane curve is moved along another curve lying in a perpendicular plane. Shells whose middle surface is of this type are often used for roof construction. This class of shells is included in one introduced earlier by the author [Ing.-Arch. 19, 228–230 (1951); these Rev. 13, 300]. He specializes his earlier analysis of the extensional ("membrane") theory to this case, obtaining relatively simple single partial differential equations for a stress function and for the normal displacement. For the case of shells with positive Gaussian curvature the author gives also an approximate means of determining displacement disturbances near the boundary when the loads and supports are not consistent with the boundary conditions appropriate to the extensional theory. The method used is outlined in an earlier work [Z. Angew. Math. Mech. 30, 370–374 (1950); these Rev. 12, 770] and is to be explained in a forthcoming book, "Theoretical elasticity," written jointly with A. E. Green. *C. Truesdell.*

Krettner, J. Anwendung der Tensorrechnung auf die Theorie der Rotationsschalen. Österreich. Ing.-Arch. 7, 246–254 (1953).

The author derives the well known equilibrium equations and approximate expressions for strains in a shell of revolution. *C. Truesdell* (Bloomington, Ind.).

Aymerich, Giuseppe. Sulla geometria delle configurazioni piane di sforzi elastici. Rend. Sem. Fac. Sci. Univ. Cagliari 22 (1952), 38–47 (1953).

The fundamental theorem of Sáenz [Neményi and Sáenz, J. Rational Mech. Anal. 1, 73–86 (1952); these Rev. 13, 703] asserts that a given congruence of plane curves determines a two to five parameter family of plane elastic systems having these curves as stress trajectories. The author constructs a parallel proof of a parallel theorem, in which the given curves are to be the lines of constant principal shear stress. With the aid of this result he is able to complete an earlier analysis concerning conjugate stress distributions. (Two distributions are said to be conjugate if their Airy functions ϕ and ψ are such that $\phi+i\psi$ is an analytic function of order two in the sense of Burgatti.) The author's theorem is to the following effect: except for a specified degenerate case, two plane stress systems having the same lines of constant shear stress and having lines of constant density orthogonal to one another are conjugate.

C. Truesdell (Bloomington, Ind.).

Gur'ev, M. F. Stress distribution in a stretched isotropic finite rectangular plate weakened by a circular opening. Dopovidi Akad. Nauk Ukrain. RSR 1953, 133–139 (1953). (Ukrainian. Russian summary)

An approximate solution of the problem of stress concentration in a stretched elliptical plate weakened by a circular hole is obtained by constructing an integral equation for the auxiliary function, with the aid of which two analytical functions in the Musheliśvili solution of the plane problem of elasticity can be calculated. The integral equation is approximated by a system of linear algebraic equations. The rectangular plate, referred to in the title of the paper, is deemed to be adequately approximated by an elliptical plate. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Radenković, Dragoš. Bending of a rectangular plate weakened by a hole. Acad. Serbe Sci. Publ. Inst. Math. 5, 133–144 (1953).

The bending of an isotropic simply supported square plate with a central hole and free edges has been considered under uniform loading. The solution for a uniformly loaded plate without hole is combined with a biharmonic function in the form of a Fourier series and the solution due to a concentrated load at the centre. The boundary conditions lead to two infinite sets of equations to evaluate the constants. Numerical results indicate that, for the ratio 1:3 for the inner and outer edges, the deflection along the inner edge is slightly greater than for the corresponding case of a plate without hole, while the distributions of moments in the two cases are entirely different. *B. R. Seth.*

Conway, H. D. Stress concentration due to elliptical holes in orthotropic plates. J. Appl. Mech. 21, 42–44 (1954).

Szabó, I. Beiträge zur Theorie der achsensymmetrisch belasteten dicken Kreisplatte insbesondere bei elastischer Lagerung. Ing.-Arch. 19, 342–354 (1951).

In an earlier paper [Ing.-Arch. 19, 128–142 (1951); these Rev. 13, 405] the author considered a thick circular disk

resting on a smooth elastic foundation, the upper surface of the disk being subjected to an axially symmetric load. In the present paper this same problem is again considered, but in the case when the contact between the cylinder and foundation is rough. The method of solution is much the same, the boundary condition on the curved surface of the cylinder being satisfied approximately by an appeal to St. Venant's principle, just as in the case of the previous problem. In the present paper the author also considers briefly a circular cylinder loaded as above, simply supported along the edge of the lower face, with the curved surface either free or built-in. *G. E. Hay* (Ann Arbor, Mich.).

Szabó, István. Die in Achsenrichtung rotationssymmetrisch belastete dicke Kreisplatte auf nachgiebiger und auf starrer Unterlage. Z. Angew. Math. Mech. **32**, 145–153 (1952). (German. English, French and Russian summaries)

The author considers a thick circular disk resting on a flat support which is either rigid or deflects vertically by an amount proportional to the normal stress acting on it. The upper face of the disk is subjected to an axially symmetric load, and on the lower face of the disk the Coulomb friction law prevails. The solutions are obtained in the same manner as in the earlier paper referred to in the review just above. When the curved edge of the disk is built-in, an exact solution is obtained; when the curved edge is free, the boundary condition on the curved edge is satisfied only approximately, through an appeal to St. Venant's principle. *G. E. Hay* (Ann Arbor, Mich.).

Szabó, István. Beiträge zur Theorie der achsensymmetrisch belasteten schweren dicken Kreisplatte. Z. Angew. Math. Mech. **32**, 359–371 (1952). (English, French and Russian summaries)

The problem treated here is the same as that covered in the preceding review, except that the body force acting on the circular cylinder is now included. The presence of this body force requires the determination of a particular solution of a certain differential equation. This solution is obtained as a polynomial. Otherwise, the procedure is quite similar to that in the paper covered in the preceding review. *G. E. Hay* (Ann Arbor, Mich.).

Kromm, A. Verallgemeinerte Theorie der Plattenstatik. Ing.-Arch. **21**, 266–286 (1953).

It is well-known that the classical Kirchhoff theory of the elastic deformations of thin plates involves fundamental differential equations whose orders are such that only two independent boundary conditions may be specified. This theory is a two-dimensional approximation to the exact three-dimensional theory. Improved approximate theories can be developed only through the rejection of some of the assumptions of the Kirchhoff theory. Papers differ mainly in the methods used for the derivation of the fundamental equations and also through the retention or neglect of certain terms connected with the initial assumptions; these assumptions are difficult to justify, and no very satisfactory conclusion has yet been reached. Such an improved theory has been developed by E. Reissner [J. Math. Physics **23**, 184–191 (1944); J. Appl. Mech. **12**, A-69–A-77 (1945); Quart. Appl. Math. **5**, 55–68 (1947); these Rev. **6**, 195; **7**, 42; **8**, 547], and is such that three independent boundary conditions may be specified. The present paper has the same objective, and the sole approximation involved in the theory developed lies in the neglect of the transverse strain. A full comparison is made between the present theory and that due to Reissner. *H. G. Hopkins* (Providence).

Hu, Hai-Chang. Small deflections of plates and beams under tension or compression by eigenfunctions of buckling problem. Acad. Sinica Science Record **5**, 69–75 (1952). (Chinese summary)

Deflections of beams and plates under the combined action of arbitrary lateral loads q and middle surface thrust S can be represented by an expansion in the eigenfunctions of the associated homogeneous buckling problem. The author presents such expansions for the uniform beam under four different combinations of conservative end conditions and for the clamped circular plate under a symmetric q. *W. Nachbar* (Seattle, Wash.).

Iacob, Caius. Sur la torsion des barres cylindriques. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **4**, 669–677 (1952). (Romanian. Russian and French summaries)

The author applies his method for the Dirichlet problem in multiply connected regions [J. Math. Pures Appl. (9) **18**, 363–383 (1939); these Rev. **1**, 283] to the solution of the torsion problem of circular rods with exterior and interior slots. A complex torsion function $F(z)$, whose real and imaginary parts are the harmonic warping and shear stress functions, is found by a contour integral along the edges of the slots after the region has been analytically extended by Schwarz's principle of inversion with respect to the boundary circle itself. This paper does not disclose the details of obtaining the functions employed in the contour integrals, except to suggest they come as a result of the usual integral of the normal derivative of Green's function and the boundary value. The following cases are treated, one radial, two opposite radial, n symmetrical radial external slots, and one internal or n internal symmetrical slots. *D. L. Holl.*

Lein, G. Die Torsionssteifigkeit von kreuzförmigen Querschnitten. Ing.-Arch. **21**, 352–364 (1953).

The torsional stiffness of rods whose section is cross-shaped, that is a central square with four projecting legs, is determined to lie between two bounds such as

$$b^4[15.760+\tfrac{8}{3}(\lambda_1+\lambda_2-6.5)]<J_t$$

$$<b^4[15.800+\tfrac{8}{3}(\lambda_1+\lambda_2-6.5)],$$

where $2b$ is the width of each leg, $(\lambda_1 b, \lambda_2 b)$ the overall length of each pair of legs, and J_t is the stiffness. It is assumed that each leg is greater in length than five times its width. Moreover there are sharp reentrant corners. When the corners have rounded fillets, the stiffness increases, the increase largely given by a change in the first term of inequality. The analysis requires considerable numerical work. *D. L. Holl* (Ames, Iowa).

* **Angles d'Auriac, P. A propos de l'équilibre des surfaces.** Comptes Rendus du Congrès des Sociétés Savantes de Paris et des Départements tenu à Grenoble en 1952, Section des Sciences, pp. 61–67. Gauthier-Villars, Paris, 1952.

Explanations and proofs of an earlier note [C. R. Acad. Sci. Paris **234**, 294–295 (1952); these Rev. **13**, 794]. The author gives the explicit solution for the sphere, and as further examples he considers torsion of the frustum of a cone and combined extension torsion of a cylinder. *C. Truesdell* (Bloomington, Ind.).

Chandra Das, Sisir. Stress concentrations around a small spherical or spheroidal inclusion on the axis of a circular cylinder in torsion. J. Appl. Mech. **21**, 83–87 (1954).

Rostovcev, N. A. Complex functions of the stresses in the axially symmetric contact problem of the theory of elasticity. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 611–614 (1953). (Russian)

A method of calculating stresses and displacements in the axially symmetric contact problem of elasticity (in the absence of friction), which does not require computing the Newtonian potential, is proposed. *I. S. Sokolnikoff.*

Grilic'kiĭ, D. V. Compression of two elastic anisotropic bodies taking account of friction forces. Dopovidi Akad. Nauk Ukrain. RSR **1953**, 122–126 (1953). (Ukrainian. Russian summary)

The note contains a solution of the plane contact problem of elasticity for two anisotropic elastic bodies, when the tangential stress along the region of contact is proportional to the normal stress. The problem is treated as one in plane deformation. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Mindlin, R. D., and Deresiewicz, H. Elastic spheres in contact under varying oblique forces. J. Appl. Mech. **20**, 327–344 (1953).

The tractions and displacements on the contact surfaces of two identical elastic spheres are investigated for the cases in which the forces applied to the spheres are varied in both magnitude and direction through a prescribed program. A concise but self-contained summary is first given of formulae derived in two earlier papers on this subject by Mindlin [J. Appl. Mech. **16**, 259–268 (1949); these Rev. **11**, 557] and by Mindlin, Mason, Osmer and Deresiewicz [Proc. 1st U. S. Nat. Congress Appl. Mech., Chicago, 1951, Amer. Soc. Mech. Engrs., New York, 1952, pp. 203–208]. For these results, the resultant force component N normal to the contact surface is assumed to remain constant, while the tangential component T is considered in successive special cases as increasing, decreasing or oscillating. The basic assumptions underlying the previous work are then formalized as "rules of procedure" under which the more complicated loadings are to be analyzed. In brief summary, these rules are: (i) The radius of the circular contact surface and the normal component of traction on it are given by the classical Hertz solution for spheres under normal loading only. (ii) The tangential traction component at any point of the contact surface cannot exceed in magnitude the product of the local normal traction component and a constant coefficient of friction; where the equality holds, slip may occur in the same sense as the tangential traction. (iii) Slip in the direction of the force causing it will progress concentrically, radially inward from the boundary of the contact surface to form an "annulus of slip". (iv) The contact surface in general comprises the annulus of slip surrounding a circular, adhered region which undergoes rigid-body tangential displacement. Previously derived formulae then give, for any particular normal traction component, the radius of the adhered portion and the tangential displacements and traction distribution. Due to the occurrence of slip and its associated energy dissipation, the changes in traction and displacement depend upon the entire past history of loading and the instantaneous relative rates of change of N and T.

Two types of problems are considered in the major portion of the paper. In the first, initial tangential compliances and other information are calculated for some seven cases of particular past histories and instantaneous rates of loading, e.g., after an initial state has been reached by first applying a normal force N and then a tangential force T, N and T are to increase at an arbitrary relative rate. In the second, a detailed history of a varying oblique force is investigated, and finally, the frictional energy loss per cycle is computed for an oscillating, oblique load.

W. Nachbar (Seattle, Wash.).

Bressan, Aldo. Sulle deformazioni dei corpi cristallini cilindrici nello schema di De Saint Venant. Rend. Sem. Mat. Univ. Padova **22**, 281–293 (1953).

The author considers the following problem. Given a cylinder of anisotropic material having three planes of elastic symmetry, does there exist a geometrically congruent cylinder of isotropic elastic material such that when subjected to a properly selected pair of statically equivalent loads of St. Venant's type the two cylinders suffer the same deformation? He finds that in general there is no solution to this problem. However, when certain special relations are satisfied by the anisotropic elasticities, he is able to exhibit solutions in the cases of simple extension, pure bending, torsion, and non-uniform bending. [Cf. a different result of this type by Lodge, Nature **169**, 926–927 (1952); these Rev. **13**, 1005.] *C. Truesdell* (Bloomington, Ind.).

Kuhelj, A. Energy criterion of elastic stability for thin shells. Acad. Serbe Sci. Publ. Inst. Math. **5**, 77–102 (1953).

The author extends to curvilinear co-ordinates the considerations of Trefftz [Z. Angew. Math. Mech. **13**, 160–165 (1933); **15**, 101–108 (1935)] concerning elastic stability. The criterion for critical load is $\delta(\Delta E-\Delta A)=0$, where ΔE is the increment in the elastic energy of the body in a small but not infinitesimal displacement from a given loaded configuration and ΔA is the work of the external forces in producing this deformation. The method is one of power series expansion, only the terms of second order in the displacements being retained, and the stress and strain increments are supposed related by the linear Hooke's law. In this way differential equations for the critical stresses are obtained (§6). The paper concludes by outlining a similar analysis for the Love theory of the bending of thin shells. The author gives a relatively simple expression for the quantity whose variation is to be zero, but he states that the actual variational equations are too complicated to be written down explicitly. *C. Truesdell.*

Hunter-Tod, J. H. The elastic stability of sandwich plates. Ministry of Supply [London], Aeronaut. Res. Council. Rep. and Memoranda **2778** (1949), 39 pp. (1953).

Die Arbeit behandelt die elastische Stabilität einfach gestützter rechteckiger geschichteter Platten mit isotroper und zwei verschiedenen Arten aeolotroper Füllung unter Druck- und Schubspannungen, letztere beide symmetrisch in bezug auf die Normale zur Plattenebene, wobei die eine Füllung Steifigkeit nur in den zur Platte senkrechten Ebenen hat, während die andere Füllung vergleichbare Steifigkeiten in allen Ebenen aufweist. Es werden Formeln für die kritischen Spannungen für ebene und schwach gebogene Platten unter Druck und ebene Platten unter Schub bei Ausknickung der ganzen Platte entwickelt; ebenfalls für die Faltung und örtliche Knickung der Stirnflächen der Platte unter Druck. Es wird festgestellt, dass innerhalb eines

grossen Gebietes die kritische Spannung für die Knickung der Platte unter Druck als unabhängig von der Form der Füllung angesehen werden kann, vorausgesetzt Symmetrie um die Normale zur Platte. Von den elastischen Konstanten der Füllung ist nur der Schubelastizitätsmodul von Bedeutung. Als ein Ergebnis wird eine einfache Erweiterung der gleichwertigen Plattentheorie von höchst verbesserter Exaktheit entwickelt, die es ermöglicht, die Gleichungen zu benutzen, welche die Platte als ein Ganzes ansehen. Wie Verf. bemerkt, zeigen neuere Versuche mit Dufaylite, dass die Annahme der Symmetrie um eine Normale zur Ebene für Füllungen von Wabenmuster ("honeycomb") nicht exakt erfüllt ist, indem der Schubelastizitätsmodul gemessen in Richtung einer Diagonale der hexagonalen Zelle sich vom Schubmodul rechtwinklig gemessen etwas unterscheidet.

R. Gran Olsson (Trondheim).

Deuker, E. A. Die strenge Lösung der Eigenwertaufgabe für den Kippstab. Ing.-Arch. **21**, 399–408 (1953).

The beam in question is cantilevered at one end and subject to a force at the other directed along a principal axis. The static deformation of the beam is desired. A variational method is used. The resulting equations are complicated, but have a simple special solution corresponding to pure bending. The author investigates the stability of this solution and concludes that it may not be stable if the stiffness in the direction of bending is too great. The situation in this case is treated by considering the second variation of the potential energy. The resulting mechanical problem is quite intricate, but the author reduces it to the problem of solving an integral equation which is in a form to be treated numerically. *E. Pinney* (Berkeley, Calif.).

Hoff, N. J., and Bruce, Victor G. Dynamic analysis of the buckling of laterally loaded flat arches. J. Math. Physics **32**, 276–288 (1954).

Bei Einführung des Parameters $e=y_{0,\max}/\rho$, wo $y_{0,\max}$ die maximale Ausbiegung im unbelasteten Zustand und ρ den Trägheitsradius des Querschnitts bezeichnen, lassen sich die Fälle des Durchschlagens seitlich belasteter flacher Bogen wie folgt einteilen. Während für $e=2$ kein Durchschlagen eintritt, kann ein solches für $2<e\leqq 4$ entstehen, aber es besteht nur eine Gleichgewichtslage im unbelasteten Zustand. Wenn $4<e<4.69$, bestehen zwei stabile Gleichgewichtslagen. In allen Fällen ($2<e<4.69$) kann die kritische Belastung demselben Diagramm entnommen werden, während für $e<4.69$ ein anderes Diagramm besteht, das ebenfalls dargestellt wurde. Für $e>4$ kann das Problem nicht als System mit nur einem Freiheitsgrad idealisiert werden, und die Gestalt der Biegelinie kann auch bei symmetrischer Belastung unsymmetrisch werden. Es ist beachtenswert, dass die endgültige Gestalt der Ausbiegung im Gleichgewicht immer symmetrisch bleibt, gleichgültig ob die Belastung vorhanden oder entfernt ist. Die kritische Last einer plötzlich aufgebrachten Last von unendlicher Dauer wurde ebenfalls berechnet. Bei dem besonders betrachteten Fall $e=8$ wurde der Wert der kritischen Last um etwa 25% niedriger gefunden als bei unendlich langsam aufgebrachter Last. Belastungen impulsiver Art (plötzlich aufgebracht, kurze Dauer) können anderseits Werte haben, die viel grösser sind als die kritischen Werte einer langsam aufgebrachten Last, ohne ein Durchschlagen auszulösen. Der kritische Wert eines Impulses der zum Durchschlagen notwendig ist, wurde ebenfalls berechnet.

R. Gran Olsson (Trondheim).

Vâlcovici, V. Détermination de la longueur critique dans le flambage des colonnes pesantes immergées dans un fluide. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **3**, 341–375 (1951). (Romanian. Russian and French summaries)

Egger, Hans. Querschwingungen von Trägern mit Feder und Zusatzmasse. Österreich. Ing.-Arch. **7**, 188–214 (1953).

The author studies vibrations of a composite system consisting of a uniform elastic beam, supported conservatively, to which at some arbitrary point a mass is attached by a linear spring. The mass is assumed to move along the spring axis which is perpendicular to the rest axis of the beam. Expressions for the frequency equation and the normal modes of free vibration are derived for the general system; two cases then treated numerically are the cantilever with mass and spring at tip and the simply-supported beam with mass and spring at the center. Charts and tables are presented for these examples giving the two or three lowest frequencies for a wide range of the possible ratios of stiffness and of mass between the two components of the system. A comparison is given with the results of D. Young [J. Appl. Mech. **15**, 65–72 (1948); these Rev. **10**, 763] for a similar composite system in which the positions of the mass and spring are reversed.

Since the roots of the frequency equation are difficult to compute for an arbitrary point of attachment of the spring, methods of obtaining approximate expressions for the fundamental frequency are discussed, and specific formulas are given for the cantilever and simply-supported beam. It is first shown that, if the beam is replaced by an equivalent mass and spring, the spring rate having its static value and the equivalent mass to be computed from the exact equations, then no single equivalent mass can be given which would give uniform and satisfactory accuracy for all points of attachment. The author then proceeds to derive approximate expressions for the fundamental frequencies by an ingenious approximation to the exact frequency equation; no analytical bounds are given for the error involved, but it is stated that these expressions are accurate to within 0.5%. Some numerical spot checks by the reviewer tended to confirm this. The reviewer would have wished to see a comparison of these results with those obtained by the more familiar method of constraining the beam to vibrate only in its lowest free mode.

The last part of this paper deals with some special cases of forced vibration of the composite system.

W. Nachbar (Seattle, Wash.).

Niordson, Frithiof I. N. Vibrations of a cylindrical tube containing flowing fluid. Trans. Roy. Inst. Tech. Stockholm no. **73**, 28 pp. (1953).

This problem arose in connection with the vibration of dam penstocks carrying water to a hydroelectric plant. A moving, slightly compressible fluid is contained within a cylindrical elastic shell. The motion of the fluid is taken as nearly rectilinear, and the deflection of the shell is assumed small so that the hydrodynamical equations can be linearized. The assumption that the velocity potential can be partially separated makes it possible to virtually "uncouple" the elastic and hydrodynamical problems. Then, while the elastic problem is theoretically solvable in trigonometric form, the algebra involved in getting the natural frequencies of oscillation is completely unwieldy. The author treats two relatively simple special cases. In the first the flow velocity

is taken as zero, and in the second the tube vibrates as a beam, i.e., it is assumed that the cross-section of the tube is unchanged in the deformed state. Calculations are made, but comparisons with experimental results, which would have been interesting, are not given. *E. Pinney.*

Tolstoy, Ivan, and Usdin, Eugene. Dispersive properties of stratified elastic and liquid media: A ray theory. Geophysics 18, 844–870 (1953).

Heelan, Patrick A. On the theory of head waves. Geophysics 18, 871–893 (1953).

Koiter, W. T. Stress-strain relations, uniqueness and variational theorems for elastic-plastic materials with a singular yield surface. Quart. Appl. Math. 11, 350–354 (1953).

This paper discusses stress vs. strain-rate relations, and uniqueness and variational theorems, for general elastic-plastic materials characterized by a singular yield surface. If a stress state corresponds to a singular point P_s of the yield surface S then the usual assumption that the plastic strain-rate vector $\dot{e}^p$ is directed along the outwards-drawn normal to S at P_s has no precise interpretation. Here it is supposed that S comprises portions of a number r of regular yield surfaces S_r, singular points of S occurring only at intersections of the S_r's. Then the modified assumption is that the usual criterion for yielding is applied for each S_r separately, this assumption always being precise and compatible with the usual one except at points P_s. At points P_s some degree of arbitrariness in the direction of $\dot{e}^p$ is explicitly recognized. The same approach has been proposed by W. Prager [sectional address, 8th Inter. Cong. Theor. Appl. Mech., Istanbul, 1952]. For example, Tresca's yield condition is singular; use of the present modified flow rule has been shown to yield extremely simple solutions of some important problems [see the paper reviewed below and references cited there at the end]. The author proves certain uniqueness and variational theorems, only slight modifications of the proofs for regular yield surfaces being required. The slip theory of Batdorf and Budiansky [NACA Tech. Note no. 1871 (1949); these Rev. 10, 648] is included in the present theory. *H. G. Hopkins* (Providence, R. I.).

Prager, William. On the use of singular yield conditions and associated flow rules. J. Appl. Mech. 20, 317–320 (1953).

Discussion is given of the plastic strain-rates at singular points on the yield surface; this follows that given previously by the author [sectional address, 8th Inter. Cong. Theor. Appl. Mech., Istanbul, 1952], and an analytical treatment along similar lines is due to W. T. Koiter [see the paper reviewed above]. The ideas are specialized to the (singular) yield condition of Tresca. The problem of the finite expansion, under internal pressure, of a circular hole in an infinite plate made of plastic-rigid material obeying the yield condition of Tresca and the associated flow rule, and exhibiting a rate of strain-hardening proportional to the rate of dissipation of mechanical energy during plastic flow, is then considered. It is shown that a simple solution exists, and this includes previous (unpublished) results of H. A. Bethe [see G. I. Taylor, Quart. J. Mech. Appl. Math. 1, 103–124 (1948); these Rev. 10, 83] for perfectly plastic material. Calculations indicate that even a small degree of strain-hardening has a marked effect on the thickness variation produced during deformation. The simplicity of the results is striking, and this is a feature of some other solutions of problems treated through the use of the same yield condition and flow rule [see Koiter, C. B. Biezeno Anniversary Volume, H. Stam, Haarlem, 1953, pp. 232–251; these Rev. 14, 1148; and Hopkins and Prager, J. Mech. Phys. Solids 2, 1–13 (1953); these Rev. 15, 270]. *H. G. Hopkins.*

Jossa, Franco. Complimenti alla teoria statica dei solidi in regime non di Hooke. Ricerca, Napoli 4, no. 3–4, 25–43 (1953).

Discussion of a one-dimensional theory in which stress is a polynomial in ϵ^{-1}, where ϵ is an undefined measure of strain. *C. Truesdell* (Bloomington, Ind.).

Grandori, Giuseppe. Strutture reticolari in equilibrio elasto-plastico. Visione energetica del problema. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 16(85), 157–169 (1952).

This paper concerns the determination of forces in a barred-structure of perfectly-plastic material. Discussion is given of various general principles, and the geometrical solution of a simple problem is given. *H. G. Hopkins.*

Todeschini, Bartolomeo. Sforzi maxwelliani e sforzi elastici nel caso piano. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 16(85), 103–108 (1952).

The author sets up the equations for plane stress in a dielectric elastic medium. He obtains solutions appropriate to a plane condenser and an infinite cylindrical condenser. *C. Truesdell* (Bloomington, Ind.).

MATHEMATICAL PHYSICS

***Morse, Philip M., and Feshbach, Herman. Methods of theoretical physics.** 2 volumes. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1953. xxii+pp. 1–997+xl; xviii+pp. 999-1978. $15.00 per volume.

The authors state the following in their preface. "This treatise is the outgrowth of a course which has been given by one or the other of the authors for the past sixteen years. The book itself has been in the process of production for more than half this time, though with numerous interruptions, major and minor. Not the least difficult problem in its development has been to arrive at a general philosophy concerning the subject matter to be included and its order of presentation." This "order-problem" of presentation has been solved by the authors as follows. Part I is divided into 8 chapters entitled: Types of fields, Equations governing fields, Fields and the variational principle, Functions of a complex variable, Ordinary differential equations, Boundary conditions and eigenfunctions, Integral equations. Part II is divided into 5 chapters entitled: Approximate methods, Solutions of Laplace's and Poisson's equations, The wave equation, Diffusion, Wave mechanics, Vector fields.

The first chapter discusses, as the authors state, the general properties of various fields and how these fields can be expressed in terms of various coordinate systems. The second chapter discusses the various types of partial differential equations which govern fields and the third chapter treats of the relation between these equations and the fundamental variational principles developed by Hamilton and

others for classic dynamics. The fourth chapter is divided in 8 sections entitled: complex numbers and variables; analytic functions; derivatives of analytic functions, Taylor and Laurent series; multivalued functions; calculus of residues; gamma and elliptic functions; asymptotic series, method of steepest descent; conformal mapping; Fourier integrals. The fifth chapter, entitled ordinary differential equations contains nevertheless the theory of partial differential equations which govern the different sorts of fields encountered in physics. But the attribute "ordinary" is justified by the predominance of the so-called separation method in this chapter which reduces partial differential equations to ordinary differential equations. Since one should not try to "squeeze a right-hand foot into a left-hand shoe", the authors develop in the sixth chapter in detail the theory of boundary conditions and eigenfunctions.

The seventh chapter is divided in five sections entitled: source points and boundary points; Green's function for steady waves; Green's function for the scalar wave equation; Green's function for diffusion; Green's function in abstract vector form. The last chapter of part I is entitled "integral equations". It starts with the integral equations of physics and their classification. The discussion of the general properties of integral equations utilizes the results for operator equations in abstract vector spaces (but without a detailed representation of the pioneering theory of D. Hilbert and E. Schmidt). Especially may be mentioned the interesting fifth section of this chapter entitled Fourier transforms and integral equations (including applications of the Laplace transform, Mellin transform, the method of Wiener and Hopf, Milne's problem and the general method of factorization).

In the first chapter of part II the authors emphasize first the application of perturbation theory to the determination of eigenvalues and eigenfunctions. In the third section of this chapter they consider perturbation methods for scattering and diffraction, including problems in which the eigenvalues form a continuous spectrum. Again integral equations are discussed (integral equation for scattering, Born approximation, Fredholm series, Schrödinger equation, WKBJ-method, penetration through a barrier). When the perturbation term is large, the perturbation methods become tedious and the physical meaning of the results is beclouded by the complexity of the expressions which are developed. In cases of this kind the authors keep ready the variational method explained in the fourth section of this chapter.

In nine chapters (1172 pages) the authors have completed the general study of fields and their behaviour. The following four chapters are devoted to a systematic application of the general technique and results to specific physical problems. The authors emphasize again that this is a book on the methods of theoretical physics, on the mathematical tools which may be used to solve problems in many branches of physics. Consequently they are concerned with the physical content of the problems only enough to show the relation to the physics but not enough to give a connected picture of the physical aspect in all its ramifications. They have, therefore, often to jump from one field of physics to another with scant regard for logical unity with regard to content, in order to show how a given technique may be used to solve problems from a large number of fields with a wide variety of content. Moreover, they are primarily concerned with the application of the more advanced techniques of computation. The authors emphasize that "the remaining chapters of this work will be rather heavy going for the casual reader, and in many places, it will be hard to see the forest for the trees". They start with the simplest equation of those under consideration, the Laplace equation and its related inhomogeneous equation, the Poisson equation (solutions in two dimensions, solutions for three dimensions). Then the wave equation is discussed (wave motion in one dimension, waves in two space-dimensions, waves in three space-dimensions, integral and variational techniques). The chapter "Diffusion" contains a sketch of the techniques of solving the simple Schrödinger equation neglecting relativistic and spin effects (solution of the diffusion equation, distribution functions for diffusion problems). In the last chapter the authors go to the calculation of purely vector fields (vector Green's theorem, dyadic Green functions, vector eigenfunctions).

This important and impressive work contains an abundance of figures of excellent quality. The authors say: "several of the figures in this work which have to do with three dimensions, are drawn for stereoscopic viewing. They may be viewed by any of the usual stereoscopic viewers or, without any paraphernalia, by relaxing one's eye-focussing muscles and allowing one eye to look at one drawing, the other at the other. Those who have neither equipment nor sufficient ocular decoupling may consider these figures as ordinary perspective drawings unnecessarily duplicated. If not benefited, they will be at least not worse off by the duplication". *M. Pinl* (Dacca).

Truesdell, C. The physical components of vectors and tensors. Z. Angew. Math. Mech. **33**, 345–356 (1953). (English, French and Russian summaries)

When we use tensor calculus in the differential equations of mathematical physics, the dependent variables are the components of tensors and these components have in general no direct physical meaning, often differing in dimensions from the physical quantities which they represent; the purpose of this paper is to bridge the gap between tensor components and physical components, the coordinates being curvilinear and in general oblique. Resolving a vector $\mathbf{v}$ in the directions of three unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ (in general oblique), so that $\mathbf{v}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$, the author defines the physical component of $\mathbf{v}$ in the direction of $\mathbf{i}$ to be a, following Ricci and Levi-Cività [Math. Ann. **54**, 125–201 (1901)]; if $ds^2=g_{ij}dx^idx^j$, the physical component of a contravariant vector λ^i in the direction of the parametric line of the coordinate x^i is $\lambda(i)=(g_{ii})^{\frac{1}{2}}\lambda^i$ (not summed). Criticising, for the most part adversely, various definitions of the physical components of tensors, he defines the physical components of a given mixed tensor a^i_j as follows. Let λ^j be a given contravariant vector and let μ^i and ν^i be defined by $\mu^i=a^i_j\lambda^j$, $\nu^i=\lambda^ia^i_j$; then the right physical components $a(ij)$ are defined by the equations $\mu(i)=a(ij)\lambda(j)$, and the left physical components $(ij)a$ analogously. A number of theorems are proved, including the following. The matrices a^i_j, a^j_i, $(ij)a$, $a(ij)$ have the same trace. If the tensor a_{ij} is symmetric, then the matrix of its left physical components is the transpose of the matrix of its right physical components. The physical dimensions of the physical components are considered; also proper values and principal axes, invariants, and tensors of order greater than 2. For orthogonal coordinates formulae are given for the physical components of the gradient of a tensor with given physical components. With minor exceptions, all the results and formulae are valid in a Riemannian space of n dimensions. [The reviewer prefers to define the physical component of a vector F_i (e.g. force) in the direction of a unit vector λ^i to be the invariant $F_i\lambda^i$, and the physical component of a tensor T_{ij} (e.g. stress),

in the direction of λ^i across a plane normal to μ^i, to be the invariant $T_{ij}\lambda^i\mu^j$. We need an additional adjective to distinguish between two different definitions of "physical component", each useful in its proper context.]

J. L. Synge (Dublin).

Popovici, Andrei. Théorie générale des constantes physiques. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **3**, 417–427 (1951). (Romanian. Russian and French summaries)

Optics, Electromagnetic Theory

Nicolau, Edmond. Relations de réciprocité et de conservation en électricité. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **4**, 739–749 (1952). (Romanian. Russian and French summaries)

L'auteur rappelle d'abord la manière dont on peut obtenir des relations de réciprocité à partir des systèmes d'équations aux dérivées partielles, linéaires et à coefficients constants; il applique ensuite la méthode aux systèmes d'équations décrivant: 1) les phénomènes électromagnétiques dans les diélectriques parfaits, homogènes et isotropes; 2) les phénomènes électriques sur les lignes de transmissions.

From the author's summary.

Crowley, T. H. On reciprocity theorems in electromagnetic theory. J. Appl. Phys. **25**, 119–120 (1954).

The author tabulates various forms of the reciprocity theorem for different source distributions in a homogeneous, isotropic, lossless medium. These forms stem from a deduction due to Lorentz [Verslagen Wis. Nat. Afd. Akad. Wetensch. Amsterdam **4**, 176–187 (1896); W. O. Schumann, Elektrische Wellen, Hanser, München, 1948, p. 332].

C. H. Papas (Pasadena, Calif.).

Keller, Herbert B. Ionospheric propagation of plane waves. Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Rep. No. **EM-56**, iii+41 pp. (1953).

The author presents a comprehensive study of wave propagation in the ionosphere, consisting of a plane horizontally stratified layer bounded on either side by free space. The dielectric tensor (depending on the magnetic field of the earth, the collisonal frequency and the density of electrons, etc.) is taken from the dissertation of van der Wyck [Delft Univ., 1946]. The longitudinal components of the field are expressed in terms of the transverse components, leading to a system of four simultaneous linear differential equations (transmission line equations), after separation of variables and plane-wave assumption (mode theory). Characteristic waves are studied by solving the equations by matrix methods [H. B. Keller and J. B. Keller, same Research Rep. No. **EM-33** (1951); these Rev. **13**, 346]. Comments on special results known from the literature; references.

C. J. Bouwkamp (Eindhoven).

Keller, Herbert B. On the electromagnetic field equations in the ionosphere. Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Rep. No. **EM-57**, i+17 pp. (1953).

First, adding slight improvements, the author summarizes van der Wyck's derivation of the dielectric tensor in ionospheric wave propagation [cf. preceding review]. Secondly, he shows the equivalence of various forms of the propagation equations, comments upon their solution by various authors and interprets Rydbeck's [Trans. Chalmers Univ. Tech. Gothenburg [Chalmers Tekniska Högskolans Handlingar] no. **74** (1948); these Rev. **10**, 252] coupled equations in particular.

C. J. Bouwkamp (Eindhoven).

Twersky, Vic. Multiple scattering of waves by planar random distributions of parallel cylinders and bosses. Division of Electromagnetic Research, Institute of Mathematical Sciences, New York University, Research Rep. No. **EM-58**, i+56 pp. (1953).

The author starts from a planar configuration of parallel cylinders and considers the scattering of a plane wave, incident at an angle with the plane of the array. Then an ensemble of scattering cylinder configurations is considered as specified by the probability distribution function for pairs of cylinders. The scattering function is obtained by an averaging process due to F. Zernike and J. Prins. The averages are represented as closed-form approximations. From these equations the average energy flux is obtained. This expression is applied to special problems, such as the scattering at grazing incidence. Finally, a plane wave incident on a configuration of bosses on a perfectly reflecting plane is considered and results are derived for the acoustic and the electromagnetic case.

M. J. O. Strutt (Zurich).

Robin, Louis, et Pereira-Gomes, Alfredo. L'antenne biconique, symétrique, d'angle quelconque. Ann. Télécommun. **8**, 382–390 (1953).

In the theory of the symmetrical biconical antenna of arbitrary angle [S. A. Schelkunoff, Proc. I. R. E. **29**, 493–521 (1941); P. D. P. Smith, J. Appl. Phys. **19**, 11–23 (1948); these Rev. **9**, 552; C. T. Tai, ibid. **19**, 1155–1160 (1948)] it is necessary for the complete determination of the internal transverse magnetic waves to find the values of n which satisfy the two equations: (1) $P_n(\cos\psi)\pm P_n(-\cos\psi)=0$, where P_n is the Legendre function of the first kind. The authors calculate the values of n which satisfy (1) for $\psi=\pi/12$, $\pi/6$, $\pi/4$, $\pi/3$, and $5\pi/12$. Then they compute the terminal and input impedances using two internal and three external transverse magnetic waves in addition to the transverse electromagnetic internal waves, i.e., the conical transmission line waves.

However, it appears that for a wide-angle conical antenna the internal transverse magnetic waves contribute only negligibly to the input impedance and, hence, can be set equal to zero from the start [C. H. Papas and R. King, Proc. I. R. E. **37**, 1269–1271 (1949); G. H. Brown and O. M. Woodward, Jr., RCA Rev. **13**, 425–452 (1952)].

C. H. Papas (Pasadena, Calif.).

Huber, A. Die Randwertaufgabe der Geoelektrik für Kugel und Zylinder. Z. Angew. Math. Mech. **33**, 382–393 (1953). (English, French and Russian summaries)

While this article contains little that is basically new either from the standpoint of mathematics or of current geophysical practice, it presents a consistent development of theory from the two-layer problem with plane interface to one with spherical and then with cylindrical boundaries. The author starts with the electrical potential field due to a point source feeding current across a plane boundary and proceeds to a discussion of the various forms the boundary conditions may take as the geometry of the interfaces is made to vary from plane to curved and to intersecting boundaries. He then proceeds to the solution by means of

zonal harmonics of nth order of the boundary-value problem for a homogeneous sphere of lesser specific resistance imbedded in an infinite medium of greater specific resistance in the three cases (1) when the source is outside the sphere, (2) when it is inside, and (3) when it is on the boundary.

Making use of various lemmas concerning cylindrical harmonics which permit development of the addition theorem in terms of modified Bessel and Hankel functions, the author presents a solution of the boundary-value problem for a cylinder of revolution imbedded in a medium of different specific resistance when the source-point is anywhere with reference to the cylinder. He devotes particular attention to the case of an electrical survey of the walls of a tunnel or mine drift of approximately circular cross-section, showing that the integrals exist and converge and indicating methods for numerical evaluation. Finally the author generalizes the theory to the extent of supplying solutions for a field point and source point outside a layered sphere or cylinder. The article is accompanied by several helpful diagrams and graphs. *J. B. Macelwane.*

Gréa, R., et Higonnet, R. Etude logique des circuits de contacts. Rev. Gén. Electricité **63**, 19–34 (1954).

Quantum Mechanics

Schwinger, Julian. The theory of quantized fields. V. Physical Review. (2) **93**, 615–628 (1954).

The author's general theory of quantized fields [Physical Review. (2) **82**, 914–927 (1951); these Rev. **13**, 520] is here applied to the description of a Dirac field in interaction with a given classical electromagnetic field which is time-dependent and strictly zero outside a finite region of space-time. Explicit expressions are obtained for the probabilities for creation of any number of pairs of Dirac particles. Assuming the electromagnetic field to have an energy-density with a finite integral over space-time, all physical quantities can be evaluated as sums of convergent series of Fredholm determinants; this use of the Fredholm theory has been made before by A. Salam and P. T. Matthews [ibid. **90**, 690–695 (1953); these Rev. **15**, 82]. Finally the author obtains simpler formulae for the particle creation probabilities by introducing a "proper-time-ordered" scattering operator Σ. In the Σ matrix the "initial" state describes the incident electrons and the outgoing positrons, the "final" state describes the incident positrons and the outgoing electrons. Simple and elegant results are obtained by this trick of allowing the positrons to "travel backwards in time", which was first invented by R. P. Feynman [ibid. **76**, 749–759 (1949)]. *F. J. Dyson* (Princeton, N. J.).

Kothari, L. S. Riesz potential and the elimination of divergences from quantum electrodynamics. Proc. Phys. Soc. Sect. A. **67**, 17–24 (1954).

The author uses the analytic continuation method of M. Riesz [Acta Math. **81**, 1–223 (1949); these Rev. **10**, 713] for the evaluation of the divergent integrals in quantum electrodynamics. Each divergent expression is replaced by a finite and well-defined function of the Riesz parameter α. As $\alpha \to 0$ these expressions tend to finite limits which involve an arbitrary constant l_0 of the dimensions of length. Thus the divergent quantities are eliminated and instead there appear finite but indefinite quantities. The method of mass and charge renormalization must still be used in order to separate the observable effects of the theory, which are independent of l_0, from the unobservable self-energy effects. In this respect the author's method is similar to that of W. Güttinger [Physical Rev. (2) **89**, 1004–1019 (1953); these Rev. **15**, 85] who interprets the divergent integrals by means of distribution theory. It appears that the Riesz method and the distribution-theory method of handling singular integrals are in some sense mathematically identical. *F. J. Dyson* (Princeton, N. J.).

Caianiello, E. R. On quantum field theory. I. Explicit solution of Dyson's equation in electrodynamics without use of Feynman graphs. Nuovo Cimento (9) **10**, 1634–1652 (1953).

The author gives a surprisingly compact algorithm for writing down the elements of Dyson's S-matrix. Let $M^{(N)}$ denote the contribution of order e^N to a scattering process in which p electrons, q positrons, and b photons are created and n electrons, m positrons, and a photons are destroyed. Denoting $n+q$ as N_0 and $a+b$ as P_0, the author shows that $M^{(N)}$ equals (to within a sign, rules for determining which are given),

$$\frac{e^N}{N!}\int \Sigma \begin{pmatrix} 1\ 2 \cdots N & V^1 \cdots V^{N_0} \\ 1\ 2 \cdots N & U^1 \cdots U^{N_0} \end{pmatrix} \cdot [1\ 2 \cdots N\ Z^1 \cdots Z^{P_0}].$$

Here $\begin{pmatrix} 1\ 2 \cdots & V^{N_0} \\ 1\ 2 \cdots & U^{N_0} \end{pmatrix}$ stands for an $(N+N_0)\times(N+N_0)$ determinant whose elements are

$$(hk) = \tfrac{1}{2}[S_F(x_h - x_k)\gamma^{\mu_h}]_{\beta_k\beta_h}, \quad (U^h h) = (hu^h) = u^h{}_{\beta_h}(x_h),$$
$$(V^h h) = (h V^h) = [v^h(x_h)\gamma^{\mu_h}]_{\beta_h}, \quad \text{and} \quad (V^h u^k) = (u^k V^h) = 0.$$

(u and v are the wave functions of initial and final electrons and positrons.) The symbol $[1\ 2 \cdots N\ Z^1 \cdots Z^{P_0}]$ stands for what the author calls a "hafnian"; a hafnian bears the same relation to a permanent which a Pfaffian bears to a determinant. The elements of the hafnian are D_F functions just as those of the determinant above are S_F functions. The hafnian algorithm seems to arise naturally in the algebra of operators obeying commutation (rather than anti-commutation) laws.

In terms of graphs, the author's formula gives all graphs, even the "unconnected" ones. This may be a disadvantage for actual computational purpose. However, the suggestion of the deeper structure in the theory which this work brings out is of great significance and a paper on this is promised. *A. Salam* (Cambridge, England).

Matthews, P. T. and Salam, Abdus. Covariant Fock equations. Proc. Roy. Soc. London. Ser. A. **221**, 128–134 (1954).

The well-known equations of Tamm and Dancoff [S. M. Dancoff, Physical Rev. (2) **78**, 382–385 (1950)] enable field-theoretical problems to be formulated and even occasionally solved, without making an explicit expansion of the solution as a power-series in the coupling constant. But the Tamm-Dancoff equations have the disadvantage of destroying the manifest covariance of a relativistic theory. The authors have successfully constructed for the first time a set of equations of the same general structure as the Tamm-Dancoff equations, but preserving manifest covariance. Whether the new equations will be simple enough to be useful in practical calculations remains to be seen.

Let Ψ be a given state of a system of interacting fields, and let Ψ_0 be the vacuum state of the same system. Let

$P_n(x_1, \cdots, x_n)$ be a product of n field operators (in Heisenberg representation) operating at n space-time points $x_1, \cdots, x_n$. Then the "wave-function" for the state Ψ is defined as consisting of certain linear combinations of the matrix elements $\langle \Psi_0 P_n(x_1, \cdots, x_n)\Psi\rangle$. The wave-function thus consists of an infinite set of functions of 4, 8, 12, $\cdots$ variables. The Matthews-Salam equations are an infinite set of coupled linear integral equations, expressing each wave-function component in terms of a finite set of other components. The equations are derived by some complicated but elementary algebra, making use of the field equations which the individual Heisenberg field-operators satisfy.

F. J. Dyson (Princeton, N. J.).

Tamor, S. The scattering of many-particle systems. Physical Rev. (2) **93**, 227–228 (1954).

For the analysis of scattering processes, involving systems of particles, an "impulse approximation" has been used [G. F. Chew, Physical Rev. (2) **84**, 710–716 (1951), where other references are given]. The essential idea of the approximation is that in the scattering, for example, of particle 1, from a composite system made up of particles 2 and 3, the scattered wave can be written as the sum of two terms, arising from the scattering of particle pairs (1, 2) and (1, 3); each weighted by a numerical factor which measures the overlap of the initial and final wave functions.

In this paper, the author shows that the possibility of writing the total scattered wave as a weighted sum of scattered waves for 2-particle scattering allows for further simplification in the formulae if Racah's coefficients are used to describe the scattering. In particular, it is possible to perform a reduction of the "magnetic sums", and the resulting expressions enable one to construct an S-matrix for the over-all scattering process in terms of S matrices for the elementary interactions. *A. Salam.*

Källén, Gunnar. Charge renormalization and the identity of Ward. Helvetica Phys. Acta **26**, 755–760 (1953).

In quantum electrodynamics there are two ways of defining the renormalization of charge. We may define the physical unit of charge so that the total charge existing in the presence of an external charge (not an electron) is equal to the external charge in the limit of low frequencies. Or we may define the unit so that the total charge existing in a one-electron state is equal to the observed electronic charge e. The two definitions are equivalent in quantum electrodynamics (but not in other field theories) in consequence of Ward's identity [Physical Rev. (2) **78**, 182 (1950); these Rev. **11**, 632]. This means that all kinds of charge, whether carried by electrons or not, are renormalized by the same factor. However, previous proofs of Ward's identity have always been formulated in terms of perturbation theory. The author here gives the first formal proof independent of perturbation theory, using his own version of quantum electrodynamics [Helvetica Phys. Acta **25**, 417–434 (1952); these Rev. **14**, 435]. The proof depends on an ingenious mathematical use of the equation of conservation of charge over a finite time-interval. *F. J. Dyson.*

Hanus, W., and Rayski, J. Vacuum polarization in a non-local electrodynamics. Acta Phys. Polonica **12**, 181–193 (1953). (Russian summary)

The authors calculate by perturbation theory the vacuum polarization produced by an external source of current, in the non-local electrodynamics of J. Rayski [Proc. Phys. Soc. Sect. A. **64**, 957–968 (1951); these Rev. **13**, 609]. They find (i) a non-gauge-invariant term corresponding to a large finite self-energy for the photon, (ii) a finite charge-renormalization term, and (iii) other finite terms representing observable polarization effects. These are exactly the same effects that have been found with other non-gauge-invariant cut-off versions of electrodynamics [R. P. Feynman, Physical Rev. (2) **76**, 769–789 (1949); these Rev. **11**, 765].

F. J. Dyson (Princeton, N. J.).

Chew, Geoffrey F. One of Schwinger's variational principles for scattering. Physical Rev. (2) **93**, 341–343 (1954).

By using one of the variation principles for treating scattering problems constructed by Schwinger [Lectures on nuclear physics, Harvard University, 1947, unpublished], the author has discussed an approximate method for investigating scattering problems with particular reference to interactions, such as the pion-nucleon, which are not diagonal in the configuration space. Three examples are treated, and plausible criteria for the validity of the approximation are proposed. *P. O. Löwdin* (Uppsala).

Moses, Harry E. The formulation of the Kohn-Hulthén variational principle in terms of the scattering operator formalism. Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Rep. No. **CX-6**, i+12 pp. (1953).

L. Hulthén [Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **14**, no. 21 (1944); these Rev. **6**, 111] has constructed a variation principle for the asymptotic phase in scattering problems, which he has later developed in a series of articles. The principle has to some degree been extended also by Kohn [Physical Rev. (2) **74**, 1763–1772 (1948); **84**, 495–501 (1951); these Rev. **13**, 807]. The present author has investigated the correspondance to this principle in the scattering operator formalism introduced by Møller, Friedrichs, Lippman, and Schwinger, and he has shown that, in this way, the variation principle takes a more abstract and general form than before. *P. O. Löwdin* (Uppsala).

Moses, Harry E. The formulation of the Kohn-Hulthén variational principle in terms of the scattering operator formalism. Physical Rev. (2) **92**, 817–821 (1953).

See the report reviewed above.

Polkinghorne, J. C. An identity for the S matrix for a finite time interval. Physical Rev. (2) **93**, 228–229 (1954).

Let $S(\tau, m_i)$ denote the S matrix for the finite time interval $t=0$ to $t=\tau$, where m_i are the masses of the fields concerned. The author shows that $S(\lambda\tau; m_i) = S(\tau; \lambda m_i)$ where λ is a positive number. The result depends on the identity $H_1(\lambda x) = \lambda^{-4} H_1^*(x)$. Here $H_1(x)$ is the interaction Hamiltonian for fields of masses m_i, and $H_1^*(x)$ the corresponding Hamiltonian for fields of masses λm_i [$x = (x_1, x_2, x_3, ict)$].

A. Salam (Cambridge, England).

Jean, Maurice. Contribution à l'étude des méthodes de la seconde quantification et de l'espace de configuration en théorie relativiste des systèmes de particules: application à la dérivation d'équations relativistes pour le deutéron. Ann. Physique (12) **8**, 338–391 (1953).

A useful, detailed account of previously announced results [C. R. Acad. Sci. Paris **232**, 1183–1185, 1290–1291, 2405–2407; **233**, 573–575, 602–604 (1951)]. The paper begins with a systematic covariant treatment of Fock space using

ideas of Becker and Leibfried for the non-relativistic case [Physical Rev. (2) 69, 34 (1946); these Rev. 7, 404] and the notations of Schwinger. Explicit formulas are given for free-field fermions and bosons in configuration and momentum representations. The interaction picture for two fermions in scalar or pseudoscalar interaction is employed to derive from field theory the Tamm-Dancoff equations and an equation which is analogous to Breit's equation in electrodynamics. The paper concludes with a discussion of the Salpeter-Bethe equation along lines similar to those of Gell-Mann and Low [ibid. 84, 350–354 (1951); these Rev. 13, 413]. *A. J. Coleman* (Toronto, Ont.).

Bodiou, Georges. Relations entre la seconde quantification, la mécanique ondulatoire dans l'espace de configuration, et les problèmes du second ordre en calcul des probabilités. J. Phys. Radium (8) 15, 39–44 (1954).

A treatment of the introduction of the creation and annihilation operators in boson fields. *I. E. Segal.*

Bodiou, Georges. Impossibilité de décomposer la fonction d'onde $\psi(M_1, \cdots, M_i, \cdots, M_N)$ d'une assemblée de fermions M_i en un produit de fonctions d'onde $\psi_i(M_i)$, individuelles. C. R. Acad. Sci. Paris 237, 1493–1495 (1953).

A complex wave-function $\psi(r_1, r_2, \cdots r_n)$ antisymmetric in the positions r_i of n particles cannot be expressed as a product $f_1(r_1)f_2(r_2)\cdots f_n(r_n)$ of single particle functions. [This rather trivial result may be proved much more simply. For example, interchange of particles 1 and 2 gives $f_1(r_1)f_2(r_2)+f_1(r_2)f_2(r_1)=0$. Multiply this by any function $g(r_1)$ orthogonal to $f_1(r_1)$ and integrate. Whence f_2 is also orthogonal to g and thus f_2 is a constant multiple of f_1, which is impossible.] *A. J. Coleman* (Toronto).

van Kampen, N. G. S matrix and causality condition. II. Nonrelativistic particles. Physical Rev. (2) 91, 1267–1276 (1953).

This continues a discussion begun in connection with the electromagnetic field [Physical Rev. (2) 89, 1072–1079 (1953); these Rev. 14, 828] of the analytic properties of the functional dependence of the inelastic scattering matrix $S(p)$ on the momentum p of the incoming particle. In the previous case, it was found that from the causality condition (essentially: physical effects are transmitted at speed less than or equal to c) alone, with no assumption about the scatterer except spherical symmetry, it followed that $S(p)$ possessed a unique analytic continuation over the whole complex p-plane with poles in the lower half of the plane as the only singularities. For the non-relativistic case there is no clear-cut velocity c, so the causality condition is reformulated: "if the ingoing wave packet is so normalized as to represent at $t=-\infty$ one incident particle, the total probability at any finite time of finding a particle outside of any sphere of radius $r \gg a$ cannot be greater than 1". Here, a is the "radius" of the scatterer.

Making use of integral transforms and some rather deep results of Nevanlinna in the theory of complex functions, the author is able to deduce from this condition, the general form of $S(p)$ in the complex plane. This form is more complicated than in the previous case and since it involves an unknown function is not of great practical use. Instead of being regular in the upper half-plane, $S(p)$ is now regular inside the first quadrant. Most authors assume, in addition to causality, the symmetry condition $S(-p)=S^*(p)$, which has been shown by Wigner to hold if the scattering is describable by a hermitian Hamiltonian. Making use of this condition, the author is able to eliminate the above unknown function and obtain a much more explicit result for S.

In addition to their mathematical interest, these papers seem to settle the question as to what can be deduced from the causality condition by itself. Of physical interest is, also, the author's conclusion that it is not possible to deduce (as Heisenberg suggested) the energies of bound states from phase-shift data. *A. J. Coleman* (Toronto, Ont.).

van Kampen, N. G. The connection between the R-matrix and the S-matrix. Revista Mexicana Fisica 2, 233–243 (1953).

A matrix R closely related to the scattering matrix S has proved useful in the theory of scattering. Wigner proved that a number of general properties of R follow from assuming that the interaction inside a non-relativistic scatterer is describable by a hermitian Hamiltonian. These properties of R are here shown to be equivalent to properties of the S matrix which follow from the causality condition.

A. J. Coleman (Toronto, Ont.).

van Kampen, N. G. Connections between the R matrix and the S matrix. Revista Mexicana Fisica 2, 222–232 (1953). (Spanish)

Spanish translation of the paper reviewed above.

Vrkljan, V. S. Ist das Diracsche Verfahren der Linearisation notwendig? Proc. Indian Acad. Sci. Sect. A. 37, 491–498 (1953).

The author poses the question whether Dirac's linearization of the wave equation for particles with non-vanishing rest-mass is the only way which leads to usable equations for the electron and the positron. He concludes that this is the case if one postulates that all the matrices used should be hermitian. He suggests, however, that one may use non-hermitian matrices in a physically meaningful way, and outlines a method of doing this. *E. Gora.*

Vrkljan, V. S. de Broglie's theory of particles of spin 1 and Ehrenfest's theorem. Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke 277, 211–219 (1950). (Serbo-Croatian)

The author shows that Ehrenfest's theorem [Z. Physik 45, 455–457 (1927)], according to which the motion of the center of mass of a probability distribution should obey Newton's laws, does not apply to the particles of spin 1 in de Broglie's theory of fusion [Mécanique ondulatoire du photon et théorie quantique des champs, Gauthier-Villars, Paris, 1949; these Rev. 10, 663]. *E. Gora.*

Vrkljan, V. S. On the magnetic moment of mesons with spin 1. Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke 277, 221–227 (1950). (Serbo-Croatian)

It is shown that the mesons of spin 1 in de Broglie's theory of fusion [Théorie générale des particules à spin, Gauthier-Villars, Paris, 1943] should possess the magnetic moment $eh/4\pi\mu_0 c$; $\mu_0/2$ is the rest-mass of the hypothetical particles whose fusion is supposed to produce the meson.

E. Gora (Providence, R. I.).

Vrkljan, Vladimir S. On the problem of extension of Dirac spinors to several components. Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke **276**, 5–37 (1949). (Serbo-Croatian)

It is shown that the matrix relations $\alpha^2=1$, $\alpha_i\alpha_j=-\alpha_j\alpha_i$, can be fulfilled with matrices of 8th order in a similar way as with the well known fourth order matrices of Dirac. The solution of the wave equations corresponding to these 8-order matrices is an 8-component spinor function. Expressions for charge-current densities are given, first for plane waves, and then for wave packets with an error function distribution. The magnetic moments of particles described by such wave functions are determined. Within the framework of this formalism, available constants can be chosen in such a way that one obtains the correct values for the magnetic moments of the neutron and the proton for particles of the respective masses and charges. In a last section it is shown that such particles may possess the correct spin value $h/4\pi$. *E. Gora* (Providence, R. I.).

Vrkljan, Vladimir S. Über das Problem der Erweiterung der Diracschen Spinore auf mehrere Komponenten. Bull. Internat. Acad. Yougoslave. Cl. Sci. Math. Phys. Tech. (N.S.) **5**, 5–7 (1952).

Summary of the paper reviewed above.

Cap, Ferdinand. Spinortheorie der Elementarteilchen. I. Grundlagen der modifizierten Theorie. Z. Naturforschung **8a**, 740–744 (1953).

Donnert, Hermann. Spinortheorie der Elementarteilchen. II. Mathematische Durchführung des Postulates der Beschreibung mit $2(2s+1)$ bzw. $4(2s+1)$ reellen Wellenfunktionen. Z. Naturforschung **8a**, 745–747 (1953).

Cap, Ferdinand. Spinortheorie der Elementarteilchen. III. Das freie Teilchen. Z. Naturforschung **8a**, 748–753 (1953).

In the first of these papers the author states that a charged elementary particle of spin s should be described by a complex wave function with $2(2s+1)$ independent components. He devotes this first paper to reviewing the existing theory of elementary particles and pointing out that the wave functions used therein do not satisfy the requirement he deems essential.

In the second paper it is proposed to ensure this requirement by requiring that a free charged particle of spin s is defined by two completely symmetric spinors $a^{(+s)}_{\dot\rho_1\cdots\dot\rho_{2s}}$ and $a^{(-s)}_{\rho_1\cdots\rho_{2s}}$ each of which satisfies the equation $(\Box-k^2)a=0$ where $\Box$ is the D'Alembertian operator and k is the reciprocal of the Compton wave length of the particle. A sequence of other wave functions is then defined in terms of these two spinors. The equivalence claimed by the author between the equation he postulates and those occurring in the usual theory is not clearly demonstrated, especially the case for $s=\frac{1}{2}$. The case of particles in external fields is not treated although it would appear that serious modifications would be needed in this case.

The third paper is concerned with the quantized theory of free charged particles as described by a pair of symmetric, s-index, two-component spinors each of which satisfies a Klein-Gordon equation. A Lagrangean for deriving these equations is given and the standard methods are used to define a current vector and a stress-energy tensor.

A. H. Taub (Zurich).

Cap, Ferdinand. A new spinor theory of elementary particles. Physical Rev. (2) **93**, 907 (1954).

Nishijima, Kazuhiko. Many-body problem in quantum field theory. Progress Theoret. Physics **10**, 549–574 (1953).

The Bethe-Salpeter wave functions of a state are regarded as covariant components of the state vector, of which the corresponding contravariant components are defined in terms of Heisenberg operators operating on the vacuum state. In this formulation Feynman kernels correspond to the metric. Integral equations for such kernels are derived by functional differentiation with respect to the external sources, and the construction of the S-matrix is discussed.

H. C. Corben (Pittsburgh, Pa.).

Baranger, M., Bethe, H. A., and Feynman, R. P. Relativistic correction to the Lamb shift. Physical Rev. (2) **92**, 482–501 (1953).

The relativistic corrections to the Lamb shift, i.e., terms of order $\alpha(Z\alpha)^5mc^2$, are calculated. For this purpose, the Lamb shift is separated into one term in which the Coulomb potential acts only once, and another term in which it acts two or more times (Sec. II). The one-potential term is shown to be equal to the expression calculated in previous papers except for corrections of order $\alpha(Z\alpha)^5$ (Sec. III), and a method is given by which these corrections could be evaluated if desired (Appendix). The many-potential term can be separated into a nonrelativistic part which is again equal to the term calculated in previous papers, and a relativistic term which can be calculated by considering the intermediate states as free (Sec. IV). The calculation of the latter term which, of course, involves the Coulomb potential exactly twice, is described in Sec. V. A correction to the vacuum polarization term which is of the same order, is evaluated in Sec. VI.

The result for the relativistic correction is 7.13 Mc/sec, and is in agreement with the result of Karplus, Klein, and Schwinger which was obtained by an independent method. The result for the complete Lamb shift has been given in a recent paper by Salpeter. The small remaining discrepancy of 0.6 Mc/sec between theory and experiment might be due to the next order relativistic correction which should be of order $\alpha(Z\alpha)^5\ln(Z\alpha)$. (Author's summary.) *A. Salam.*

Yamamoto, Takao. Analytical representation of general spin. Progress Theoret. Physics **10**, 579 (1953).

Baumann, Kurt. Die relativistische Beschreibung von Bindungszuständen. Acta Physica Austriaca **8**, 4–15 (1953).

In dieser Arbeit wird eine Zusammenfassung der Fortschritte gegeben, die in den letzten Jahren in einer relativistischen Behandlung von Bindungszuständen erzielt wurden. Die Zusammenhänge zwischen den verschiedenen Methoden werden aufgezeigt, soweit es in diesem Gebiet, dessen Erforschung noch im Fluss und in raschem Fortschreiten begriffin ist, möglich erscheint.

Zusammenfassung des Verfassers.

Roussopoulos, Paul N. Sur la diffusion élastique d'une particule par un système complexe. C. R. Acad. Sci. Paris **238**, 213–215 (1954).

Thermodynamics, Statistical Mechanics

Klein, Martin J. A note on Wild's solution of the Boltzmann equation. Proc. Cambridge Philos. Soc. **50**, 293–297 (1954).

E. Wild [same Proc. **47**, 602–609 (1951); these Rev. **13**, 195] has given a formal but rather involved procedure for obtaining series solutions of the Boltzmann equation in the kinetic theory of gases for certain situations including that in which the molecular distribution is initially homogeneous in space and molecules repel one another with an inverse fifth-power force (Maxwell molecules). As an illustration of the application of this procedure, the author derives the known exponential decay of the second velocity moments for a gas satisfying the above conditions. He concludes that the derivation of this particular result by Wild's method is more complicated than by older methods. The author's conclusion, however, should not necessarily decrease the value of this calculation nor the expected future value of Wild's methods. *G. Newell* (Providence, R. I.).

Nanda, V. S. Bose-Einstein condensation and the partition theory of numbers. Proc. Nat. Inst. Sci. India **19**, 681–690 (1953).

The author investigates the statistical properties of an ideal Bose-Einstein gas in one dimension. Using asymptotic formulae from the theory of partitions, he obtains accurate information about the occupation numbers of the low-lying states, which show some minor departures from the behavior estimated from the usual statistical approximations. *F. J. Dyson* (Princeton, N. J.).

Gotusso, Guido. Sulle equazioni dei fluidi in meccanica aleatoria. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **16(85)**, 383–390 (1952).

Referring to previous work of Udeschini [Rend. Sem. Mat. Fis. Milano **20**, 54–80 (1950); these Rev. **12**, 620; cf. also J. Bass, Groupement Française Dévelop. Rech. Aéronaut. Rap. Tech. no. **28** (1946), §84; these Rev. **11**, 699], the author wishes to establish moment differential equations for a general stochastic mechanics. Letting ϕ and ψ be the probability densities for location $\mathbf{x}$ alone and for simultaneous location and velocity $\mathbf{v}$, he postulates

$$\phi(\mathbf{x}, t+\tau)=\int_{-\infty}^{\infty}\phi(\mathbf{x}-\mathbf{v}\tau, t)\psi(\mathbf{v}, \mathbf{x}-\mathbf{v}\tau, t)d\mathbf{v}.$$

Expanding each side in series of powers of τ and equating coefficients yields a sequence of equations whose general member is

$$\frac{\partial^n\phi}{\partial t^n}+\int_{-\infty}^{\infty}\left\{\frac{\partial(\phi\psi)}{\partial x_i}v^i\right\}^{(n)}d\mathbf{v}=0,$$

where $\{\ \}^{(n)}$ is a symbolic power. The case $n=1$ yields normalization of ψ; $n=2$ yields an equation identical in form to the continuity equation, with ϕ replacing the mass density; $n=3$ yields an equation identical in form with Cauchy's equations of motion; while $n=4, 5, \cdots$ yield differential equations containing higher derivatives. [This expansion process is very close to the now usual moment sequence for the Liouville equation. Cf. Born and Green, Proc. Roy. Soc. London. Ser. A. **188**, 10–18 (1946); these Rev. **9**, 402, and many later works on the kinetic theory of liquids.] *C. Truesdell* (Bloomington, Ind.).

Porod, Günther, und Smola, Friedrich. Die statistische Gestalt von Fadenmolekülen. I. Die Richtungsverteilung im Persistenzmodell. Acta Physica Austriaca **8**, 63–88 (1953).

In connection with the statistical study of chain molecules, the distribution of the angle between first and nth chain components is derived in terms of the distribution law for the angle between successive components, by the familiar expansion in spherical harmonics. Examples are given. All considerations, although quite elementary, are given in full detail. *L. Van Hove* (Princeton, N. J.).

Porod, Günther. Die statistische Gestalt von Fadenmolekülen. II. Die allgemeine Lösung des speziellen Irrflugproblems. Acta Physica Austriaca **8**, 181–190 (1953).

A new explicit expression is derived for the distribution of distances after n steps in a random walk problem with steps of constant length and isotropically distributed directions. *L. Van Hove* (Princeton, N. J.).

Frank, F. C., and Nicholas, J. F. Stable dislocations in the common crystal lattices. Philos. Mag. (7) **44**, 1213–1235 (1953).

The Burgers vector $\mathbf{b}$ of a perfect dislocation in a crystal can be any lattice vector, but stability requires that $\mathbf{b}$ should not be expressible as $\mathbf{b}_1+\mathbf{b}_2$ where $b^2>b_1^2+b_2^2$ (b is the length of $\mathbf{b}$). The stability of these dislocations is investigated for simple cubic, face-centered cubic, hexagonal closed packed, and body-centered lattices. In isolated imperfect dislocations a finite set of Burgers vectors corresponding to the edges of dislocation may be introduced and every Burgers vector is a sum of one of these and a lattice vector. Analogous stability considerations hold here. The more general situation involving complex imperfect dislocations is discussed by means of examples. *F. J. Murray.*

Kanô, Kenzi, and Naya, Shigeo. Antiferromagnetism. The Kagomé Ising net. Progress Theoret. Physics **10**, 158–172 (1953).

The authors solve the Onsager problem [Physical Rev. (2) **65**, 117–149 (1944); these Rev. **5**, 280] for a "Kagomé" net which consists of alternate hexagons and triangles. They use the Kaufman spinor technique [ibid. **76**, 1232–1243 (1949)] with an extra limiting procedure. The entropy at $T=0$, for negative binding energy, is evaluated also by a limiting process and the result is similar to that obtained by G. H. Wannier for the triangular lattice [ibid. **79**, 357–364 (1950); these Rev. **12**, 576], inasmuch as $S\neq 0$ and no reference order exists. *F. J. Murray.*

TREATISES ON FUNCTIONAL ANALYSIS

Two important works on functional analysis are described below. The author of each was later elected to the Presidency of the American Mathematical Society. There is little or no overlapping between the two volumes and each has established itself firmly as a standard reference work and text in its portion of the field.

LINEAR TRANSFORMATIONS IN HILBERT SPACE AND THEIR APPLICATIONS TO ANALYSIS. By M H. Stone. American Mathematical Society Colloquium Publications, vol. 15, 1932, viii+622 pp.; reprinted 1946, 1948, 1951 List Price $8.00

"The first chapter gives the definition of the abstract Hilbert space and its main properties, as well as various realizations of the space. . . .

Transformations are introduced in Chapter 2. A careful study is made of the notions of domain, range, extension, inverse, closure, continuity, linearity, and adjointness in connection with transformations. . . .

Chapter 3 illustrates these concepts by examples from the theory of infinite matrices, integral operators including Fourier and Hankel transforms, and differential operators.

In Chapter 4 the main problem is the study of the inverse of the transformation $T_l = T - lI$, where l is a complex number. This leads to the spectrum [point, continuous, and residual spectra in general] and to the resolvent R_l of T.

The detailed study of self-adjoint transformations is taken up in Chapter 5. . . .

Chapter 6 deals with operational calculus. The basic problem is to give a sense to the notion of a function of a transformation. . . .

On the basis of this operational calculus the author solves in Chapter 7 the difficult problem of unitary equivalence of two self-adjoint transformations. . . .

In Chapter 8, the basic idea is that of permutability of two transformations which leads to various complications in the unbounded case. It contains further a study of unitary and normal . . . transformations, the theory of which is closely allied to the self-adjoint case.

The general symmetric transformations are taken up in Chapter 9.

Chapter 10 is devoted to applications. . . . The author starts with the theory of unbounded integral operators, . . . which is a special case and also a forerunner of the general theory. This is followed by the theory of differential operators in which the author gives a very thorough and systematic discussion of differential systems of the first and second order. . . . The last topic treated is the theory of Jacobi matrices which includes the moment problem for infinite intervals and the theory of continued fractions. . . .

To sum up, Professor Stone has done us a real service by providing a clear, systematic, and readable account of this important theory. . . . The book is not always easy reading, but the author is fair to the reader: nothing essential is withheld, the terminology is clearly defined and strictly adhered to, there are enough facts provided for the imagination to feed on, but no loose statements. . . ."

From review by E. Hille, Bull. Amer. Math. Soc., November 1934.

FUNCTIONAL ANALYSIS AND SEMI-GROUPS. By Einar Hille. American Mathematical Society Colloquium Publications, vol. 31, 1948, xii+528 pp.
List Price $7.50

"This is a very interesting and important treatise, embodying several books as it were, the largest among the latter being the one actually devoted to semi-groups themselves. The treatise might be described as a 'cours d'analyse' for the study of vector-valued functions, Banach norms and spectral theory; but it also contains such 'classical' material as Laplace integrals, Fourier series, Hermite expansions, real functions in Euclidean spaces, and so on."

From review by S. Bochner, Bull. Amer. Math. Soc., May 1949.

"The book is divided into three parts plus an appendix. My desire to give a practically self-contained presentation of the theory required the inclusion of an elaborate introduction to modern functional analysis with special emphasis on function theory in Banach spaces and algebras. This occupies Part One of the book and the Appendix; these portions can be read separately from the rest and may be used as a text in a course on operator theory. It is possible to cover most of the material in these six chapters in two terms.

The analytical theory of one-parameter semi-groups occupies Part Two while Part Three deals with the applications to analysis. The latter include such varied topics as trigonometric series and integrals, summability, fractional integration, stochastic theory, and the problem of Cauchy for partial differential equations. In the general theory the reader will also find an alternate approach to ergodic theory. All semi-groups studied in this treatise are referred to a normed topology; semi-groups without topology figure in a few places but no details are given."

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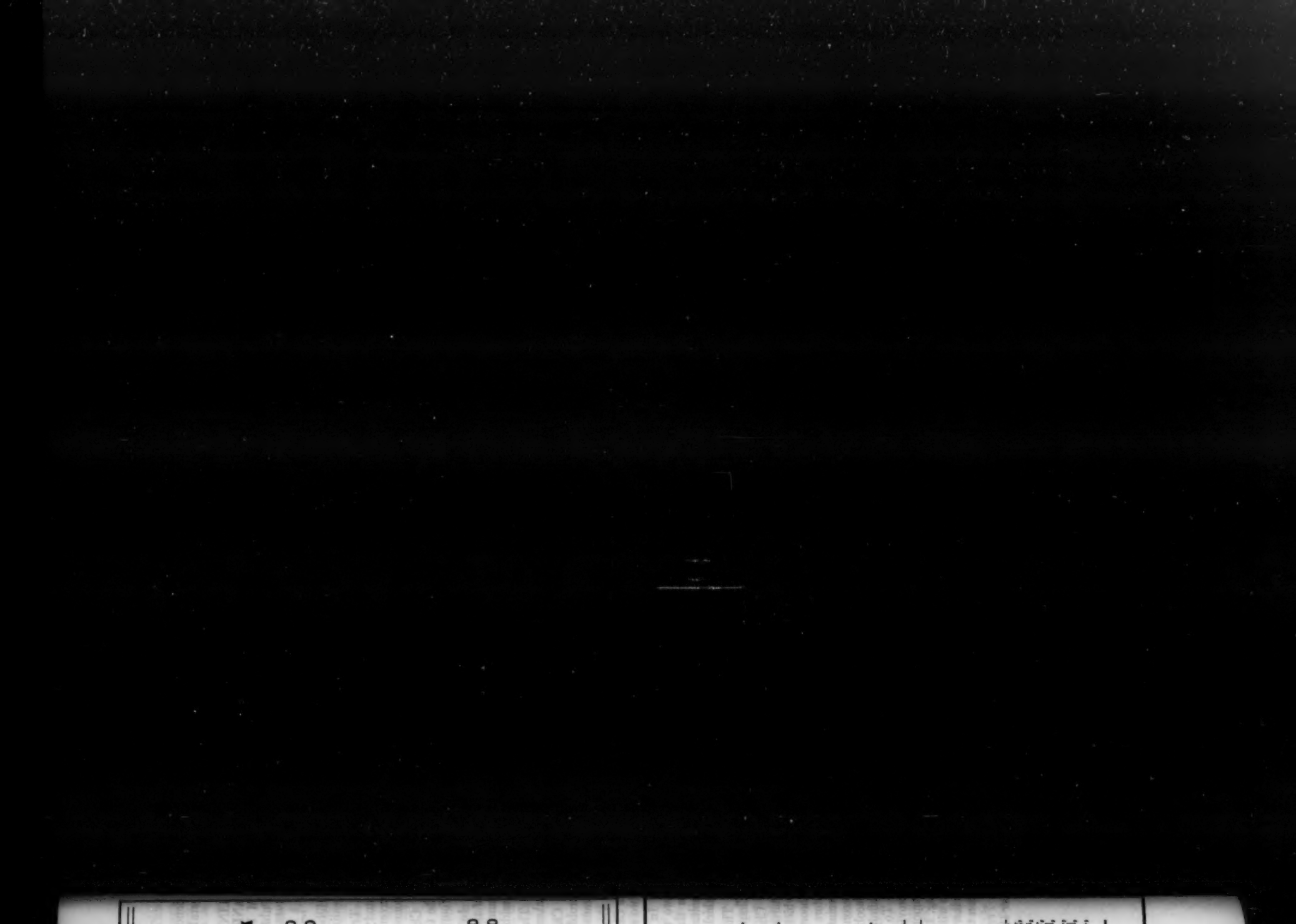
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